\[ E_0 = (\hat{x} - j\hat{y})E_0 \]

It is apparent that a reversal of the sense of rotation may be obtained by a 180-degree phase shift applied either to the \( x \) component or to the \( y \) component of the electric field.

**Elliptical Polarization.** A somewhat more general example arises when the \( x \) and \( y \) components of the electric field differ in amplitude. Assuming again that the \( y \) component leads the \( x \) component by 90 degrees, such a field may be represented by the complex vector

\[ E_0 = \hat{x}A + j\hat{y}B \]

in which \( A \) and \( B \) are positive real constants. The corresponding time-varying field is given by

\[ \mathbf{E}(0, t) = \hat{x}A \cos \omega t - \hat{y}B \sin \omega t \]

The components of the time-varying field are

\[ E_x = A \cos \omega t \]
\[ E_y = -B \sin \omega t \]

from which it is evident that

\[ \frac{E_x^2}{A^2} + \frac{E_y^2}{B^2} = 1 \]

Thus the endpoint of the \( \mathbf{E}(0, t) \) vector traces out an ellipse and the wave is said to be **elliptically polarized.** Inspection of the equations indicates that the sense of polarization is again left-handed.

Elliptical polarization is in fact the most general form of polarization. The polarization is completely specified by the orientation and axial ratio of the polarization ellipse and by the sense in which the endpoint of the electric field vector moves around the ellipse. Various methods for representing the polarization of a wave are covered in chap. 12.

**5.08 Direction Cosines.** Sometimes it is necessary to write the expression for a plane wave that is traveling in some arbitrary direction with respect to a fixed set of axes. This is most conveniently done in terms of the direction cosines of the normal to the plane of the wave. By definition of a uniform plane wave the equiphase surfaces are planes. Thus in the expression

\[ \mathbf{E}(x) = E_0 e^{-j\beta x} \]

for a wave traveling in the \( x \) direction, the planes of constant phase are given by the equation

\[ x = \text{a constant} \]

For a plane wave traveling in some arbitrary direction, say the s
direction, it is necessary to replace \( x \) with an expression that, when put equal to a constant, gives the equiphase surfaces. The equation of a plane is given by

\[
\hat{n} \cdot \mathbf{r} = \text{a constant}
\]

where \( \mathbf{r} \) is the radius vector from the origin to any point \( P \) on the plane and \( \hat{n} \) is the unit vector normal to the plane (sometimes called the wave normal). That this is so can be seen from Fig. 5-5, in which a plane perpendicular to the unit vector \( \hat{n} \) is seen from its side, thus appearing as the line \( F-F \). The dot product \( \hat{n} \cdot \mathbf{r} \) is the projection of the radius vector \( \mathbf{r} \) along the normal to the plane, and it is apparent that this will have the constant value \( OM \) for all points on the plane. Now the dot product of two vectors is a scalar equal to the sum of the products of the components of the vectors along the axes of the co-ordinate system. Therefore

\[
\hat{n} \cdot \mathbf{r} = x \cos A + y \cos B + z \cos C
\]

where \( x, y, z \), are the components of the vector \( \mathbf{r} \), and \( \cos A, \cos B, \cos C \) are the components of the unit vector \( \hat{n} \) along the \( x, y, \) and \( z \) axes. \( A, B \) and \( C \) are the angles that the unit vector \( \hat{n} \) makes with the positive \( x, y, \) and \( z \) axes, respectively. Their cosines are termed the direction cosines or direction components of the vector.

The equation of a plane wave traveling in the direction \( \hat{n} \), normal to the planes of constant phase, can now be written as

\[
\mathbf{E} (\mathbf{r}) = E_0 e^{-j \beta \hat{n} \cdot \mathbf{r}}
\]

\[
= E_0 e^{-j \beta (x \cos A + y \cos B + z \cos C)}
\]

(5-63)

In time-varying form and assuming \( E_0 = E_r + jE_i \), this becomes

\[
\mathbf{E} (\mathbf{r}, t) = \text{Re} \{E_0 e^{-j(\beta \hat{n} \cdot \mathbf{r} - \omega t)}\}
\]

\[
= E_r \cos (\beta \hat{n} \cdot \mathbf{r} - \omega t) + E_i \sin (\beta \hat{n} \cdot \mathbf{r} - \omega t)
\]

(5-64)

Wavelength and Phase Velocity. The uniform plane wave expressions considered so far all can be broken up into functions of the form

\[ e^{-j \beta \hat{n} \cdot \mathbf{r}} \]
where $h$ is some real constant and $u$ represents distance measured along a straight line. Referring back to the original discussion on wavelength and phase velocity, we see that these quantities can be stated in terms of $h$ and $\omega$ for the given direction $\hat{u}$. The wavelength in the $\hat{u}$ direction is given by

$$\lambda_u = \frac{2\pi}{h}$$  \hspace{1cm} (5-65)

and the phase velocity in the $\hat{u}$ direction is given by

$$v_u = \frac{\omega}{h}$$  \hspace{1cm} (5-66)

With this information, we can determine the wavelength and phase velocity in any direction.

For the uniform plane wave already discussed, the wavelength and phase velocity in the direction of the wave normal $\hat{n}$ are given by

$$\lambda = \frac{2\pi}{\beta} \quad \text{and} \quad v = \frac{\omega}{\beta}$$

If we look at the uniform plane wave expression in direction cosine form, it is easy to determine the wavelengths and phase velocities in the directions of the co-ordinate axes by comparison with the $e^{-j\lambda \hat{u}}$ form discussed above. Thus we have for the $x$ direction

$$\lambda_x = \frac{2\pi}{\beta \cos A} = \frac{\lambda}{\cos A} \quad \text{and} \quad v_x = \frac{\omega}{\beta \cos A} = \frac{v}{\cos A}$$

As long as the angle $A$ is not zero, both the wavelength and the phase velocity measured along the $x$ axis are greater than when measured along the wave normal. Similar statements hold for the $y$ and $z$ directions. These relations between the velocities and wavelengths in the various directions are shown more clearly in Fig. 5-6, which shows

Figure 5-6. Relations between wavelengths and velocities in different directions.
successive crests of an incident wave intersecting the \( y \) and \( z \) axes. For small angles of \( \theta \) it is seen that the velocity \( v_y \), with which a crest moves along the \( y \) axis, becomes very great, approaching infinity as \( \theta \) approaches zero.

**PART II—REFLECTION AND REFRACTION OF PLANE WAVES**

**5.09 Reflection by a Perfect Conductor—Normal Incidence.** When an electromagnetic wave traveling in one medium impinges upon a second medium having a different dielectric constant, permeability, or conductivity, the wave in general will be partially transmitted and partially reflected. In the case of a plane wave in air incident normally upon the surface of a perfect conductor, the wave is entirely reflected. For fields that vary with time neither \( E \) nor \( H \) can exist within a perfect conductor so that none of the energy of the incident wave can be transmitted. Since there can be no loss within a perfect conductor, none of the energy is absorbed. As a result the amplitudes of \( E \) and \( H \) in the reflected wave are the same as in the incident wave, and the only difference is in the direction of power flow. If the expression for the electric field of the incident wave is

\[ E_i e^{-j\beta z} \]

and the surface of the perfect conductor is taken to be the \( x = 0 \) plane as shown in Fig. 5-7, the expression for the reflected wave will be

\[ E_r e^{j\beta x} \]

where \( E_r \) must be determined from the boundary conditions. Inasmuch as the tangential component of \( E \) must be continuous across the boundary and \( E \) is zero within the conductor, the tangential component of \( E \) just outside the conductor must also be zero. This requires that the sum of the electric field strengths in the initial and reflected waves add to give zero resultant field strength in the plane \( x = 0 \). Therefore

\[ E_r = -E_i \]

The amplitude of the reflected electric field strength is equal to that of the initial electric field strength, but its phase has been reversed on reflection.

The resultant electric field strength at any point a distance \(-x\) from the \( x = 0 \) plane will be the sum of the field strengths of the
incident and reflected waves at that point and will be given by
\[ E_T(x) = E_i e^{-j\beta x} + E_r e^{j\beta x} \]
\[ = E_i (e^{-j\beta x} - e^{j\beta x}) \]
\[ = -2jE_i \sin \beta x \]  \hspace{1cm} (5-67)
\[ E_T(x, t) = \text{Re} \{-2jE_i \sin \beta x e^{j\omega t}\} \]

If \( E_i \) is chosen to be real,
\[ E_T(x, t) = 2E_i \sin \beta x \sin \omega t \]  \hspace{1cm} (5-67a)

Equation (67) shows that the incident and reflected waves combine to produce a standing wave, which does not progress. The magnitude of the electric field varies sinusoidally with distance from the reflecting plane. It is zero at the surface and at multiples of half wavelength from the surface. It has a maximum value of twice the electric field strength of the incident wave at distances from the surface that are odd multiples of a quarter wavelength.

Inasmuch as the boundary conditions require that the electric field strength be reversed in phase on reflection in order to produce zero resultant field at the surface, it follows that the magnetic field strength
must be reflected without reversal of phase. If both magnetic and electric field strengths were reversed, there would be no reversal of direction of energy propagation, which is required in this case. Therefore, the phase of the reflected magnetic field strength $H_t$ is the same* as that of the incident magnetic field strength $H_i$ at the surface of reflection $x = 0$. The expression for the resultant magnetic field will be

$$H_r(x) = H_i e^{-j\beta x} + H_i e^{+j\beta x}$$

$$= H_i (e^{-j\beta x} + e^{+j\beta x})$$

$$= 2H_i \cos \beta x$$

(5-68)

$H_i$ is real since it is in phase with $E_i$.

$$\tilde{H}_r(x, t) = \text{Re} [H_r(x)e^{j\omega t}]$$

$$= 2H_i \cos \beta x \cos \omega t$$

(5-68a)

The resultant magnetic field strength $H$ also has a standing-wave distribution. In this case, however, it has maximum value at the surface of the conductor and at multiples of a half wavelength from the surface, whereas the zero points occur at odd multiples of a quarter wavelength from the surface. From the boundary conditions for $H$ it follows that there must be a surface current of $J_s$ amperes per meter, such that $J_s = H_r(x = 0)$.

Since $E_i$ and $H_i$ were in time phase in the incident plane wave, a comparison of (67) and (68) shows that $E_r$ and $H_r$ are 90 degrees out of time phase because of the factor $j$ in (67). This is as it should be, for it indicates no average flow of power. This is the case when the energy transmitted in the forward direction is equalled by that reflected back.

That $E_r$ and $H_r$ are 90 degrees apart in time phase can be seen more clearly by rewriting (67) and (68). Replacing $-j$ by its equivalent $e^{-j\pi/2}$ and combining this with the $e^{j\omega t}$ term to give $e^{j[\omega t - (\pi/2)]}$, eq. (67) becomes

$$E_r(x, t) = \text{Re} \{2E_i \sin \beta x \ e^{-j\pi/2} \ e^{j\omega t}\}$$

$$= 2E_i \sin \beta x \cos (\omega t - \pi/2)$$

(5-69)

*An alternative way of arriving at this same result is from a consideration of current flow in the conductor. If it is assumed for the incident wave, which is traveling to the right in the positive $x$ direction, that $E_i$ is in the positive $y$ direction and $H_i$ is in the positive $z$ direction (it will be seen later that the direction of energy propagation is always the direction of the vector $E \times H$), the current flow in the conductor will be in the same direction as the incident electric field, that is, in the positive $y$ direction. This current flow produces an electric field $-E_r$ to oppose the incident field (Lenz's law) and produces a magnetic field, which is shown by application of the right-hand rule to be in the positive $z$ direction. Therefore the magnetic field of the reflected wave has the same direction as in the incident wave.