Boundary Conditions: Once we solve the Poisson's Eq., we will get the potential due to a given charge distribution but with along with a number of arbitrary constants. The values of this constants have to be determined by a number of specified conditions on the fields in the region of interest. Generally, these conditions are in the form of the value of potential or electric fields at certain boundary surfaces of the region. The boundary conditions can also be certain specified surface charge density or linear charge densities.

Boundary condition on the Electric fields: A surface divides a region in two parts, say 1 and 2. Let $\vec{E}_1$ be the electric field near the surface in region 1 and $\vec{E}_2$ be the electric field in region 2. Consider a Gaussian surface of area $\Delta a$ and thickness $2\varepsilon$. Where, $\varepsilon \to 0$. 
The total flux of the electric field from the Gaussian surface is
\[ \mathbf{E}_1 \cdot \mathbf{n} \, \text{da} - \mathbf{E}_2 \cdot \mathbf{n} \, \text{da} \]
Equation (1).

It must be noted here that the flux through the curved cylindrical surface is not zero since the electric field is not necessarily perpendicular to the surface. Secondly, the minus sign in Eq (1) above is because the normal \( \mathbf{n} \) on the two sides of the surface are oppositely directed.

Now if the surface has a surface charge density \( \sigma \), then the total charge enclosed by the Gaussian surface is \( \sigma \, \text{da} \). Then by Gauss' Law,
\[ \mathbf{E}_1 \cdot \mathbf{n} \, \text{da} - \mathbf{E}_2 \cdot \mathbf{n} \, \text{da} = \frac{\sigma \, \text{da}}{\varepsilon_0} \]
\[ \mathbf{E}_1 \cdot \mathbf{n} - \mathbf{E}_2 \cdot \mathbf{n} = \frac{\sigma}{\varepsilon_0} \]
\[ \mathbf{E}_1 - \mathbf{E}_2 = \frac{\sigma}{\varepsilon_0} \]
Equation (2).

Eq (2) specifies the boundary condition on the perpendicular components of the electric field on the two sides.

If these the surface charge density at a place is 0 then:
\[ \mathbf{E}_1 = \mathbf{E}_2 \]

Actually, Eq (2) is a form of the differential form of Gauss' Law.

The condition on the far tangential component of the electric field comes from the property that \( \mathbf{E} \times \mathbf{E} = 0 \).

Consider a rectangular loop as shown. The tangential elements are of length \( l \) and very near the surface. The parts of the loop that project the surface are negligibly small so the integral becomes.
\[ \phi E_{\parallel} \cdot d\ell = \int (\vec{E} \times \vec{E}) \cdot \hat{n} \cdot da = 0. \]

This is true for any arbitrary curve \( C \) and \( C_2 \) are tangential to the surface and since they lie close to the surface, we have \( C_2 = -C_1 \):

\[ \int_{C_1} \vec{E}_1 \cdot d\ell + \int_{C_2} \vec{E}_2 \cdot d\ell = 0. \]

\[ \int_{C_1} \vec{E}_1 \cdot d\ell - \int_{C_1} \vec{E}_2 \cdot d\ell = 0. \]

\[ \int_{C_1} \vec{E}_1 \cdot d\ell = \int_{C_2} \vec{E}_2 \cdot d\ell. \]

Since this is true for any arbitrary loop we select and hence any arbitrary curve \( C \), we have:

\[ \vec{E}_{1\parallel} = \vec{E}_{2\parallel} \quad (3) \]

where \( E_{1\parallel} \) and \( E_{2\parallel} \) are the parallel or tangential components of the electric fields to the surface.

We see that the boundary condition \( (3) \) is independent of the surface charge density.

**Boundary condition on the Electric potential:**

Let \( V_1 \) be the potential at a point very near to the surface in region 1 and \( V_2 \) in region 2:

\[ V_2 - V_1 = \int_{C} \vec{E} \cdot d\ell. \]

Since \( E \) is finite in the two regions, though it may be discontinuous, the integral on

\[ V_1 = V_2 \] near the surface. R.H.S \( \to 0 \) as \( a \to b \) as \( a \) and \( b \) approach the surface.

Therefore \( V_2 \to V_1 \)
Q: A sphere has a uniform charge density \( \sigma \) over its surface. Find the potential inside and outside the sphere.

\[ \nabla^2 V_{\text{out}} = 0 \]

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V_{\text{out}}}{\partial r} \right) = 0 \]

\[ V_{\text{out}} = \frac{C_1}{r} + C_2 \]

Similarly, \( \nabla^2 V_{\text{in}} = 0 \) \( \Rightarrow \) \( V_{\text{in}} = \frac{d_1}{r} + d_2 \)

\[ E_{\text{in}} = -\nabla V_{\text{in}} = -\frac{d_1}{r^2} \]

Applying Gauss's law over a spherical surface inside, we get \( -\frac{d_1}{r^2} \times 4\pi r^2 = 0 \) \( \Rightarrow \) \( d_1 = 0 \).

If we demand that the potential at \( \infty \) be 0, then \( C_2 = 0 \).

\[ V_{\text{out}} = \frac{C_1}{r} \]

\[ E_{\text{out}} = -\nabla V_{\text{out}} = -\frac{C_1}{r^2} \]

At \( r = R \), we have the boundary condition

\[ E_{\text{out}} \parallel - E_{\text{in}} \parallel = \frac{\sigma}{\varepsilon_0} \Rightarrow -\frac{C_1}{R^2} = \frac{\sigma}{\varepsilon_0}. \]

\[ C_1 = -\frac{\sigma R^2}{\varepsilon_0} \]

\[ V_{\text{out}} = \frac{\sigma R}{\varepsilon_0} \]
\[ V_{in} = d_2 \]

At the surface of the sphere we have:

\[ V_{out} \bigg|_R = V_{in} \bigg|_R \]

\[ \frac{\sigma R^2}{\varepsilon_0} = d_2 \Rightarrow d_2 = \frac{\sigma R}{\varepsilon_0} \]

\[ V_{in} = \frac{\sigma R}{\varepsilon_0} \]

From this we get the potential everywhere.

From this we can calculate the electric field.

\[ E_{out} = -\nabla V_{out} = \frac{\sigma R^2}{\varepsilon_0 \cdot r^2} \hat{r} \]

\[ E_{in} = 0 \]

Verify that this is indeed the right electric field obtained by other methods.