Gauss's law:

Consider a point charge \( q \) at the origin. Consider a volume \( V \) surrounded by a surface \( S \).

The flux of \( \vec{E} \) over the surface \( S \) is given by:

\[
\Phi_{\vec{E}} = \int_S \vec{E} \cdot d\vec{A}.
\]

If we increase the charge \( q \) then the electric field will increase in the same proportion. So we may say the flux of the electric field across the closed surface is proportional to the charge \( q \) at the center. In fact we need not keep the charge \( q \) at the center, but it can be kept anywhere within the surface \( S \). The constant of proportionality is \( \frac{q}{\varepsilon_0} \).

\[
\Phi_{\vec{E}} = \frac{q}{\varepsilon_0}.
\]

We can have several charges \( q_1, q_2, \ldots, q_n \) within the surface \( S \). The total electric field due to all these charges is \( \vec{E} + \vec{E}_2 + \ldots + \vec{E}_n = \vec{E} \). It is clear that

\[
\Phi_{\vec{E}} = \frac{1}{\varepsilon_0} \int_S \left( q_1 + q_2 + \ldots + q_n \right) = \frac{\text{total charge}}{\varepsilon_0}.
\]

This is the statement of the Gauss's law.
Now let us go back to the case of

If we have a continuous charge distribution \(\sigma(\mathbf{r})\) then

the Gauss's law takes the form

\[
\oint_S \mathbf{E} \cdot \mathbf{n} \, d\mathbf{a} = \frac{1}{\varepsilon_0} \int_V \sigma(\mathbf{r}) \, dV.
\]

By divergence theorem,

\[
\int_V \nabla \cdot \mathbf{E} \, dV = \frac{1}{\varepsilon_0} \int_V \sigma(\mathbf{r}) \, dV.
\]

This is true for charge distribution over any closed volume \(V\), so the integrands can be equated.

\[
\nabla \cdot \mathbf{E} = \frac{\sigma(\mathbf{r})}{\varepsilon_0}
\]

This is the differential form of the Gauss's law.

Eg: Find the electric field inside and outside a uniformly charged sphere of radius \(a\).

Let the uniform charge density be \(\sigma_0\).

\[
\sigma(\mathbf{r}) = \begin{cases} 
\sigma_0 & : 0 < r \leq a \\
0 & : r > a
\end{cases}
\]

To find the electric field for outside the sphere, \(r > a\), consider a spherical surface of radius \(r\). Due to spherical symmetry of the problem, the magnitude of the electric field \(\mathbf{E}\) is same over the surface of this sphere and directed radially outward. Let \(\mathbf{E}(r)\) be the magnitude.
of the electric field. The flux of this field over the sphere of radius \( r \) is

\[ \int \vec{E} \cdot d\vec{A} = E(0) \cdot 4\pi r^2. \]

According to Gauss' law, the flux is equal to \( \frac{q}{\varepsilon_0} \) where \( q \) is the charge enclosed by the sphere. (We call this the imaginary sphere, the Gaussian sphere.)

\[ q = \frac{4}{3} \int_{V} \rho dV = 80 \cdot \frac{4}{3} \pi a^3. \]

So by \( E(r) \cdot 4\pi r^2 = \frac{80}{\varepsilon_0} \cdot \frac{4}{3} \pi r^3 \)

\[ E(r) = \frac{80}{3 \varepsilon_0} \cdot \frac{a^3}{r^2}. \]

So at a point \( r > a \) the electric field outside the sphere is

\[ E(\infty) = \frac{80}{3 \varepsilon_0} \cdot \frac{a^3}{r^2}. \]

To calculate the electric field inside the sphere we consider a Gaussian surface, with a sphere with \( r < a \)

Then the total charge inside the sphere is

\[ q = \frac{4}{3} \pi r^3 \cdot 80. \]

By Gauss' law then

\[ E(r) \cdot 4\pi r^2 = \frac{80}{\varepsilon_0} \cdot \frac{4}{3} \pi r^3. \]

\[ E(r) = \frac{80}{3 \varepsilon_0} \cdot \frac{r^3}{r^2}. \]

\[ E(\infty) = \frac{80}{3 \varepsilon_0} \cdot \frac{a^3}{r^2}. \]

\[ E(\infty) = \frac{80}{3 \varepsilon_0} \cdot \frac{a^3}{r^2}. \] (inside).
Now we verify the differential form of Gauss's law.

For outside the charge configuration:

\[ \vec{\nabla} \cdot \vec{E}(r) = \frac{8\pi\alpha^3}{3\varepsilon_0} \cdot \vec{\nabla} \left( \frac{\hat{z}}{r^2} \right) \]

We have seen that this divergence is 0 for \( r > a \).

Hence:

\[ \vec{\nabla} \cdot \vec{E}(r) = 0 \quad \text{for} \quad r > a. \]

Inside the sphere:

\[ \vec{\nabla} \cdot \vec{E}(r) = \frac{8\pi}{3\varepsilon_0} \vec{\nabla} \cdot \vec{\nabla} = \frac{8\pi}{3\varepsilon_0} \cdot 3 = \frac{8\pi}{\varepsilon_0}. \]

So this is consistent with the differential form of Gauss's law; inside as well as outside the sphere.
Eq: A cylindrically symmetric charge distribution is given with 
\( S(s) = ks \) from \( s = 0 \) to \( s = R \). Find the electric field inside the cylinder of radius \( R \) and outside it.

By cylindrical symmetry, \( E \) is along \( \hat{z} \) everywhere.

Inside the charge distribution,

\[
\oint \vec{E} \cdot d\vec{A} = E_s(s) \cdot 2\pi s h = \frac{\text{charge}}{\varepsilon_0}.
\]

The flux through the upper and lower flat surfaces of the
Gaussian cylinder is zero since \( \vec{E} \) is perpendicular to the normal to this surface.

\[
\text{Gauss's Law: } \quad \int_V \nabla \cdot \vec{E} \, dV = \int_S \vec{E} \cdot d\vec{A} = \frac{\text{charge}}{\varepsilon_0} = \int_0^R \int_0^\infty \int_0^{2\pi} k s \cdot \vec{s} \, ds \, dz \, d\phi
\]

\[
k h \cdot 2\pi \int_0^{2\pi} \int_0^\infty s^2 ds = \frac{2\pi k h s^3}{3}
\]

\[
E_s(s) \cdot 2\pi s h = \frac{1}{\varepsilon_0} \cdot \frac{2\pi k h s^3}{3}
\]

\[
E_s(s) = \frac{2\pi k s^2}{3\varepsilon_0}
\]

\[
E_{\text{in}} = \frac{ks^2}{2\varepsilon_0}
\]

\( \propto s^2 \) (proportional to \( s^2 \))

Now we calculate electric field outside.

\[
E_s \times 2\pi s h = \frac{1}{\varepsilon_0} \cdot \frac{2\pi k h R^3}{3} \quad \text{(The charge density exists only up to } s = R)
\]

\[
E_s = \frac{1}{3\varepsilon_0} \cdot \frac{kr^2}{s}
\]

\[
E_{\text{out}} = \frac{kr^2}{3\varepsilon_0} \quad \text{[proportional to } \frac{1}{s} \text{]}
\]
we can calculate. This even using the differential form of Gauss’s law.

by symmetry of the problem, we have the component $E_\phi = E_z = 0$.
So only $E_s$ is present. Using expression for divergence in

cylindrical coordinates, we get

$$V = \frac{1}{s^2} E_s = \frac{1}{5}$$

$$\frac{1}{s^2} \frac{d}{ds} (s E_s) = \frac{1}{5} k \Phi$$

$$\Rightarrow E_s = \frac{k s^2}{5}$$

(inside the radius $R$)

outside the radius $R$ we have.

$$V=E=0$$

$$\frac{1}{s^2} \frac{d}{ds} (s E_s) = 0 \Rightarrow s E_s = C \Rightarrow E_s = \frac{C}{s}$$

where $C$ is some constant.

We demand $E_s$ to be continuous at $s = R$ (not always true

as we will see later)

$$\Rightarrow \frac{C}{R} = \frac{k R^2}{3} \Rightarrow C = \frac{k R^2}{3}$$

$$\hat{E}_\text{out} = \frac{k R^2}{3} \frac{1}{s} \hat{s}$$

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* In fact when we solve the differential Eqn in $s$ we will get

$$E_s = \frac{k s^2}{5} + C_1$$

where $C_1$ is an arbitrary constant.

Let us check by calculating the divergence of this field

at $s = 0$. This we will do carefully.

$$V.E = \lim_{s \to 0} \frac{1}{s^2} (E_s \cdot 2 \pi s h) = \lim_{s \to 0} \left( \frac{2 k s}{3} + \frac{2 C_1}{3 s} \right)$$

If $C_1 \neq 0$ then $V.E \to \infty$ as $s \to 0$ which is not

consistent with the given charge density at $s = 0$. 
Gauss' Law Eq. 2.

Electric field due to an infinite plane of charge with uniform surface charge density \( \sigma \).

Let us find the \( \vec{E} \) at a distance \( r \) from the plane.

Due to symmetry, \( \vec{E} \) is directed perpendicular to the plane.

The Gaussian surface we consider is a cylinder as shown in the figure, whose length is \( 2a \) and extends symmetrically on both sides of the charged plane.

There is no flux from the side walls of the cylinder since \( \vec{E} \) is zero orthogonal to the normals to the surface. However, the flux from the lid of the cylinder is:

\[
E \cdot d\alpha + E \cdot d\beta = 2E d\alpha.
\]

By Gauss's Law:

\[
2E d\alpha = \frac{\sigma}{\epsilon_0},
\]

\[
E = \frac{\sigma}{2 \epsilon_0}.
\]

\( \therefore \vec{E} = \frac{\sigma}{2 \epsilon_0} \hat{n} \) where \( \hat{n} \) is the normal to the plane.

Note that \( \hat{n} \) is directed opposite on the two sides of the plane. This electric field is independent of distance \( r \) and extends up to \( \infty \). Of course, this is practically not possible.

This is a hypothetical problem.