Advanced Wireless Communications (Lecture 8: Introduction to OFDM)

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Acknowledgment

- The material in this lecture is based on the book “Fundamentals of Wireless Communications” by David Tse (University of California, Berkeley) and Pramod Viswanath (University of Illinois, Urbana Champaign).
- Another good book used was “Fundamentals of Digital Communication” by Upamanyu Madhow.
Motivation for OFDM

- Orthogonal frequency division multiplexing (OFDM) is a baseband technology that uses **multiple orthogonal carriers** for transmission.
- OFDM helps counter the **frequency selective** nature of a wireless channel.
  - Recall from our earlier lectures that frequency selectivity of a wireless channel causes inter symbol interference (ISI).
- Before the advent of OFDM, data transmission used to happen on single carriers.
  - This type of transmission required a large tap complex equalizer to mitigate ISI.
- OFDM on the other hand frees the receiver from the complex equalization process.
- Further OFDM can be implemented in the **digital domain**, thereby allowing cost-effective implementation.
  - The above mentioned advantages are indeed key to its adoption in a multitude of standards namely: IEEE802.11 (WLAN), IEEE802.16 (Wi-Max) and digital video broadcasting (DVB) standard in Europe.
Why transmission via complex exponentials

- OFDM pursues transmission over multiple complex exponential functions, which are orthogonal.
- The reason why complex exponentials are chosen is because, they are indeed eigenfunctions (an input to linear time invariant (LTI) system is said to be eigenfunction if output is a scaled version of the input) of LTI systems as shown below:
  - Suppose $h(t)$ is the impulse response of an LTI system. Now if an input $x(t) = e^{j2\pi ft}$ is passed through it, then the output $y(t)$ would be

$$
y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j2\pi f(t-\tau)} d\tau
= e^{j2\pi ft} \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f \tau} d\tau
= H(f) e^{j2\pi ft}
$$

- From (1), we conclude that the complex exponentials are eigenfunctions of LTI systems.
Why transmission via complex exponentials (ctd...)

- Note the similarity between eigenfunction and eigenvector that is encountered in linear algebra.

- Another important property of complex exponentials is because complex exponentials at different frequencies are orthogonal to each other. This property is demonstrated below:

  - Suppose if \( e^{j2\pi f_1 t} \) and \( e^{j2\pi f_2 t} \) are two complex exponentials at \( f_1 \) and \( f_2 \) respectively. Then their inner product

    \[
    < e^{j2\pi f_1 t}, e^{j2\pi f_2 t} > = \int_{-\infty}^{\infty} e^{j2\pi f_1 t} (e^{j2\pi f_2 t})^* dt \quad (* \text{ — complex conjugate})
    \]

    \[
    = \int_{-\infty}^{\infty} e^{j2\pi f_1 t} (e^{-j2\pi f_2 t}) dt
    \]

    \[
    = \delta(f_2 - f_1) = 0, \ (\text{for } f_1 \neq f_2)
    \]

- Note:
  - The orthogonal property of complex exponentials holds over infinite time horizon.
  - Indeed for such a case, the selection of frequencies also can be arbitrary.
Modification of properties of complex exponentials for OFDM Systems

- However, in an OFDM system, the transmission is over a finite symbol interval.
- Therefore, we need to figure out as how the two aforementioned properties of complex exponentials (namely eigenfunction and orthogonality) must be modified.
- In practice since, an OFDM system employs a discrete set of subcarriers over a symbol interval of finite length $T$. Therefore, the transmitted complex baseband form $u(t)$ is given by

$$u(t) = \sum_{n=0}^{N-1} B[n]p_n(t) \tag{3}$$

where $B[n]$ is the symbol transmitted and $p_n(t) = e^{j2\pi f_n t}I_{[0,T]}$ is the modulating signal, $f_n$ is the frequency of the $n^{th}$ subcarrier and $I_A$ is the indicator function of set $A$.
- Note that $P_n(f) = T\text{sinc}((f - f_n)T)$. Note that $P_n(f) = 0$ if $|f - f_n| = k/T$, where $k$ is an integer.
Modification of properties of complex exponentials for OFDM Systems (ctd...)

• Now, if $T > T_d$, $T_d$ is the multipath delay spread, then $1/T << B_{\text{Co}}$, $B_{\text{Co}}$ is the coherence bandwidth. In this case, the gain seen by $P_n(f)$ is roughly constant.

• The above observation ensures that the eigenfunction property is roughly preserved. i.e., if $P_n(f)$ goes through a channel with transfer function $G_C(f)$, then the output $Q_n(f)$

$$Q_n(f) = G_C(f)P_n(f) \approx G_C(f_n)P_n(f).$$  \hspace{1cm} (4)

• The orthogonality of different subcarriers holds over an interval $T$, if they are spaced apart by an integer multiple of $1/T$:

$$\int_0^T e^{j2\pi f_n t} e^{-j2\pi f_m t} = \frac{e^{j2\pi (f_n-f_m)T} - 1}{j2\pi (f_n-f_m)} = 0, \text{ for } (f_n-f_m)T = \text{nonzero integer}$$  \hspace{1cm} (5)
From the above description, it is clear that if we choose $f_n = n/T$ in the term $p_n(t) = e^{j2\pi f_n t}I_{[0,T]}$ of (3), we can re-write $u(t)$ as

$$u(t) = \sum_{n=0}^{N-1} B[n]p_n(t) = \sum_{n=0}^{N-1} B[n]e^{j2\pi nt/T}I_{[0,T]} \quad (6)$$

Now when the signal $u(t)$ goes through the channel, the $n^{th}$ subcarrier at frequency $f_n$ sees a gain of $G_C(f_n)$.

Therefore, the receiver sees a noisy version of $G_C(f_n)B_n$ when demodulating the $n^{th}$ subcarrier.

Thus the effect of channel can be undone separately for each subcarrier.

If we sample the signal in (6) at rate $1/T_s$, where $T_s = T/N$, we obtain

$$u(kT_s) = b(k) = \sum_{n=0}^{N-1} B[n]e^{j2\pi nk/N}. \quad (7)$$
Of course, (7) is nothing but the inverse DFT of the symbol sequence 
\{B[n]\}.

In practice, the samples \(b[k]\) can be efficiently generated from \{B[n]\} using inverse fast Fourier transform (I-FFT) method.

Once \(b[k]\) is generated, \(u(t)\) in (6) can be obtained using D/A conversion.

The symbols \(B[n]\) can be obtained using the following relationship:

\[
B[n] = \frac{1}{N} \sum_{k=0}^{N-1} b[k] e^{-j2\pi nk/N}.
\] (8)

Thus (8) can be done using the FFT method at the receiver.

The (noiseless) received OFDM signal is modeled as

\[
v(t) = \sum_{k=0}^{N-1} b[k] p(t - kT_s)
\] (9)
The sampled version of the received signal in (9) can be obtained as

\[ v[m] = \sum_{k=0}^{N-1} b[k] h[m - k] \]  

(10)

where \( \{h[l] = p(lT_s)\} \) is the effective discrete-time channel length \( L \) which is assumed to be less than \( N \).

Therefore (10) can be written as

\[ v[m] = \sum_{l=0}^{L-1} h[l] b[m - l] \]  

(11)

For most practical cases \( h[l] = 0 \) for \( l < 0 \) and \( l \geq L \).

The (10) is nothing but the discrete linear convolution between the channel impulse response \( h[n] \) and the transmitted sequence \( b[n] \)

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Motivation for Cyclic Prefix in an OFDM System

- Now if the receiver takes a DFT of (11), then the output may not be of the form \( v[n] = H[n]B[n] \) (which is what is desired), where \( H[n] \) and \( B[n] \) are DFTs of the sequences \( h(n) \) and \( b(n) \) respectively.
  - This is because the DFT of the linear convolution is not equal to the product of the individual DFT's. i.e., if \( x(n) \) and \( h(n) \) are two finite sequences then
    \[
    \text{DFT} \ (x(n) \ast h(n)) \neq X[n]H[n] \tag{12}
    \]
    where \( \ast \) denotes linear convolution, \( X[n] \) and \( H[n] \) are DFT of the sequences \( x(n) \) and \( h(n) \) respectively.

- However the DFT of cyclic convolution is equal to the product of the individual DFT's. i.e., if we define cyclic convolution (denoted as \( \otimes \)) between \( x(n) \) and \( h(n) \) as
  \[
  x(n) \otimes y(n) = \sum_{m=0}^{N-1} h(m)x((n - m) \mod N), \ 0 \leq n \leq N - 1.
  \]
  then \( \text{DFT} \ (x(n) \otimes h(n)) \neq X[n]H[n] \tag{13} \)
Motivation for Cyclic Prefix in OFDM Systems (ctd...)

- In general, the linear convolution of two sequences $x(n)$ (of length $L$) and $h(n)$ (of length $P$) has a maximum length equal to $(L + P - 1)$.
- Therefore for circular convolution to emulate linear convolution exactly, the circular convolution must have length at least equal to $L + P - 1$.
- Now if we want the product $X[n]H[n]$ to represent DFT of $x(n) * H(n)$, then the DFTs that we compute must be of that length to avoid time aliasing. (i.e., if we use $N$-point DFT, then $N \geq L + P - 1$). This implies that both $x(n)$ and $h(n)$ must be of length $L + P - 1$. Therefore both $x(n)$ and $y(n)$ are padded with zeros, so that their overall length is $L + P - 1$, this is called as zero padding.
Motivation for Cyclic Prefix in OFDM Systems (ctd...)

- Therefore in the current case if we want to obtain $H[n]B[n]$, then we need to emulate linear convolution using cyclic convolution.
- For the emulation to be perfect $h(n)$ and $b(n)$ must at least be of length $N + L - 1$, where $N$ is the length of $b(n)$ and $L$ is the length of $h(n)$.
- Therefore, we add an extra $L - 1$ bits to the data $b(n)$ before transmission.
- These $(L - 1)$ bits are prefixed to the data $b(n)$, hence are referred to as cyclic prefix.
  - These $(L - 1)$ bits can be zeros. But adding zeros can lead to harmonics that are difficult to filter from the overall signal.
  - Therefore $(L - 1)$ data symbols of the sequence $\{b[k]\}$ are cyclically rotated and added to the beginning of the OFDM symbol, which explains the name “cyclic prefix”.

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CT537 - Advanced Wireless Communications – 13 / 41
Motivation for Cyclic Prefix in OFDM Systems (ctd...)

- With the addition of cyclic prefix, it can be easily verified that the linear convolution will be equal to the circular convolution.
- To see this let the cyclic prefixed \((N + L - 1)\) OFDM symbol \(d\) be
  \[
  d \triangleq \begin{bmatrix}
  b[N - L + 1], & b[N - L + 2], & \cdots, & b[N - 1], & b[0], & b[1], & \cdots, & b[N - L],
  \\
  =d[1], & =d[2], & =d[L-1], & =d[L], & =d[L+1], & =d[N]
  \end{bmatrix}
  \]

\[
\begin{align*}
  & b[N - L + 1], \cdots, b[N - 1] \\
  & =d[N+1], \quad =d[N+L-1]
  \end{align*}
\]

Now the noiseless received signal (expressed using linear convolution) is given as

\[
v[m] = h[m] * d[m] = \sum_{l=0}^{L-1} h[l]d[m - l], \quad m = 1, 2, \cdots, N + L - 1. \quad (14)
\]
- The noiseless received signal expressed using cyclic convolution can be written as
  \[
  \tilde{v}[m] = h[m] \otimes d[m] = \sum_{l=0}^{L-1} h[l]d[(m - l) \text{ modulo } N], \quad m = 1, 2, \cdots, N+L-1. \quad (15)
  \]
Motivation for Cyclic Prefix in OFDM Systems (ctd...)

• Now it can be verified that
\[ v[L], \ldots, v[N + L - 1] = \tilde{v}[L], \ldots, \tilde{v}[N + L - 1]. \]
  - For example it can be verified that
  - Similarly
    \[
    \[
    \]
    \[
    = b[N - 1] + b[N - 2].
    \]
    \[
    = b[N - L + 1] + b[N - L].
    \]
  - Thus from the above two sub-items we notice that
    \[ v[N + L - 1] = \tilde{v}[N + L - 1]. \]
• Note that at the output the cyclic prefix is removed before taking the DFT at the receiver. That is why, in above terms, we consider the components from $L$ to $N + L - 1$ in $v$ (and $\tilde{v}$).

• Thus by adding a cyclic prefix we ensure that the output of DFT operation at the receiver is equal to $H[n]B[n]$.
  ○ This implies that by adding a cyclic prefix, we are maintaining the eigenfunction property of complex exponentials, even for the finite symbol duration case.
The cyclic convolution $\mathbf{v} = \mathbf{h} \otimes \mathbf{b}$ between the channel $\{h_m\}$ and the data $\{b_m\}$ (Note: here we denote $h[m]$ and $b[m]$ as $h_m$ and $b_m$ respectively) can be expressed in terms of a linear transformation as follows:

$$\mathbf{v} = \mathbf{C}_{N \times N} \mathbf{b}_{N \times 1}$$

where

$$\mathbf{C} = \begin{bmatrix}
    h_0 & 0 & \cdots & 0 & h_{L-1} & h_{L-2} & \cdots & h_1 \\
    h_1 & h_0 & 0 & \cdots & 0 & h_{L-1} & \cdots & h_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
    h_{L-1} & h_{L-2} & \cdots & h_0 & 0 & 0 & \cdots & 0 \\
    0 & h_{L-1} & h_{L-2} & \cdots & h_1 & h_0 & 0 & \cdots \\
    \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & 0 & \cdots & 0 & h_{L-1} & h_{L-2} & \cdots & h_0
\end{bmatrix}$$

is a circulant matrix i.e., the rows are cyclic shifts of each other.

The $N \times 1$ received vector $\mathbf{y}$ can now be written as

$$\mathbf{y} = \underbrace{\mathbf{C} \mathbf{b}}_{\mathbf{v}} + \mathbf{w}$$

(Note: Here we denote $h[m]$ and $b[m]$ as $h_m$ and $b_m$ respectively.)
Cyclic Convolution using Circulant Matrices (ctd...)

- The matrix $C$ can be eigen decomposed as follows:

$$ C = U^H \Lambda U, \quad (19) $$

where $U$ is a unitary matrix such that $U^H U = I$, $I$ denotes an identity matrix and $\Lambda$ is a diagonal matrix that contains the eigen values of the matrix $C$.

- The eigen values of any circulant matrix $C$ are equal to the DFT of its first row elements $^1$. Therefore

$$ \Lambda_{nn} = \tilde{h}_n = \sum_{k=0}^{N-1} c_k e^{-j2\pi nk/N} \quad n = 0, \cdots, N - 1 \quad (20) $$

where $c_k$'s are the first row elements of matrix $C$.

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$^1$Toeplitz and Circulant Matrices: A Review by Robert M. Gray. Available at http://ee.stanford.edu/gray
• Using (19), we can write (18) as

\[ y = U^{-1} \Lambda U b + w \quad \text{(since } U^H = U^{-1}) \]  
(21)

\[ \tilde{y} = \Lambda \tilde{b} + \tilde{w} \]  
(22)

where \( \tilde{y} = Uy, \tilde{b} = Ub \) and \( \tilde{w} = Uw \). Note that \( \tilde{w} \) has the same distribution as that of \( w \).

• The above representation suggests a natural rotation at the input and output to convert the channel to a set of non-interfering channels with no ISI.

• The above interpretation is depicted in the figure shown in next slide.
Cyclic Convolution using Circulant Matrices (ctd...)

- Note that in the figure below \( \tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n \) (\( n = 0, 1, \cdots, N - 1 \)), where \( \tilde{y}_n, \tilde{b}_n, \tilde{w}_n \) correspond to the elements of \( \tilde{y}, \tilde{b}, \tilde{w} \) respectively (defined under (22)) and \( \tilde{h}_n \) is defined as in (20).
Power Spectral Density of an OFDM System

- The figure below shows the power spectral density (PSD) of an OFDM system as a function of the normalized frequency \((= fT/N_c)\). Here we assume \(N_c = 64\).

- If the loss due to cyclic prefix is very low (which is possible if the number of subcarriers \(N_c\) is large), then OFDM achieves Nyquist-rate signaling.
Power Spectral Density of an OFDM System (ctd...)

- However $N_C$ cannot be chosen arbitrarily large because it increases
  - *Peak to average power ratio (PAPR)* of the OFDM signal.
  - *Inter-carrier interference (ICI)*
- We will discuss both these issues in the following slides.
PAPR of OFDM Systems

- One of the main problems with implementation of OFDM is its peak to average power ratio (PAPR) is very high.

- By definition PAPR is defined as follows:
  
  - For continuous-time signal
    
    \[ \text{PAPR} \triangleq \frac{\max_{t} |s(t)|^2}{\mathbb{E}_t [|s(t)|^2]}, \] (23)

    where \( s(t) \) is the continuous-time transmit signal over one OFDM symbol and \( \mathbb{E}_t \) denotes mathematical expectation with respect to time \( t \).

  - The definition of PAPR based on continuous-time signal is in general hard to compute. Therefore, we define PAPR analogously in discrete-domain as
    
    \[ \text{PAPR} \triangleq \frac{\max_{l} |s[l]|^2}{\mathbb{E}_l [|s[l]|^2]}, \] (24)

    where \( s[n] \) is the discrete-time transmit signal i.e., sampled version of the continuous-time signal \( s(t) \).

- The PAPR defined in (24) lends itself to a simple characteriztion.
PAPR of OFDM Systems (ctd...)

- We will now try to get an insight into the PAPR of an OFDM system.
- To this end, consider the following passband transmit signal over one OFDM symbol (of length say $T$)

$$s(t) = \Re \left[ \frac{1}{\sqrt{N_c}} \sum_{i=0}^{n-1} \tilde{d}_i \exp \left( j2\pi \left( f_c + \frac{IN_c}{nT} \right) t \right) \right], \quad t \in [0, T]. \quad (25)$$

where the operator $\Re$ denotes the real part of the complex signal, $f_c$ denotes the carrier frequency, $n$ represents the active number of subcarriers and $\tilde{d}_i$ denotes the (QPSK, QAM) modulated symbols to be transmitted.

- We now compute the average power $P_{av}$ of the signal defined in (25)

$$P_{av} \triangleq \frac{1}{T} \int_0^T s^2(t) dt \quad (26)$$
PAPR of OFDM Systems (ctd...)

- Expressing \( s(t) = \frac{z + z^*}{2} \), where \( z \) is the argument of the “real” operator in (25) and \( * \) is the complex conjugate operator, we obtain

\[
P_{av} = \frac{1}{4N_c} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \left[ \text{term 1} + \text{term 2} + \text{term 3} + \text{term 4} \right] \tag{27}
\]

where in the terms below \( M \triangleq \frac{N_c}{n} \)

\[
\text{term 1} = \frac{1}{T} \int_0^T \tilde{d}_i \tilde{d}_k \exp \left( j2\pi \left( 2f_c + \left\{ \frac{(i + k)M}{T} \right\} t \right) \right) dt \tag{28}
\]

\[
\text{term 2} = \frac{1}{T} \int_0^T \tilde{d}_i \tilde{d}_k^* \exp \left( j2\pi \left( \frac{(i - k)M}{T} \right) t \right) dt \tag{29}
\]

\[
\text{term 3} = \frac{1}{T} \int_0^T \tilde{d}_i^* \tilde{d}_k \exp \left( j2\pi \left( \frac{(i - k)M}{T} \right) t \right) dt \tag{30}
\]

\[
\text{term 4} = \frac{1}{T} \int_0^T \tilde{d}_i^* \tilde{d}_k^* \exp \left( -j2\pi \left( 2f_c + \left\{ \frac{(i + k)M}{T} \right\} t \right) \right) dt \tag{31}
\]
PAPR of OFDM Systems (ctd...)

- The magnitude of the term 1 in (28) can be upper bound by the quantity \( \frac{|\tilde{d}_i\tilde{d}_k|}{2\pi f_c T} \). Since \( \zeta \triangleq f_c T \) is of the order of \( 10^6 \), therefore the upper bound \( \frac{|\tilde{d}_i\tilde{d}_k|}{2\pi f_c T} \) is close to zero and therefore term 1 can be ignored for all \( i, k \) respectively.

- By a similar argument the term 4 for any \( i, k \) can also be ignored.

- The term 3 and term 4 in (29) and (30) can be shown to be equal to \( \tilde{d}_i\tilde{d}_k^* \delta (i - k) \) and \( \tilde{d}_i^*\tilde{d}_k \delta (i - k) \) respectively, where \( \delta (.) \) denotes the Dirac delta function.

- Using the above three points we can simplify (27) as

\[
P_{av} = \frac{1}{4N_c} \sum_{i=0}^{n-1} 2|\tilde{d}_i|^2 = \frac{1}{2N_c} \sum_{i=0}^{n-1} |\tilde{d}_i|^2
\]

(32)

- If all of \( d_i \) have same energy i.e., if they are chosen from an equal energy constellation such as PSK, then \( |\tilde{d}_i|^2 \) is same for all \( i \) i.e., \( |\tilde{d}_i|^2 \) equal to \( E_s \), a constant. Here we assume \( E_s = 1 \). Therefore

\[
P_{av} = \frac{n}{2N_c}
\]

(33)
PAPR of OFDM Systems (ctd...)

- To compute the PAPR based on discrete-time signal defined in (24), we need to determine $\max_l |s[l]|^2$, for this we need to obtain the samples of the continuous signal $\{s[l]\}$, where $s[l] = s[l/W]$.
- To obtain the samples $\{s[l]\}$, we sample $s(t)$ defined in (25) at a rate equal to $1/W$ i.e.,

$$s(t) = \Re \left[ \frac{1}{\sqrt{N_c}} \sum_{i=0}^{n-1} \tilde{d}_i \exp \left( j2\pi \left( f_c + \frac{iN_c}{nT} \right) t \right) \right]$$ (from (25))

$$= \Re \left[ \left\{ \frac{1}{\sqrt{N_c}} \sum_{i=0}^{n-1} \tilde{d}_i \exp \left( j2\pi iN_c t \right) \right\} \exp (j2\pi f_c t) \right]$$ (34)

$$= \Re [d(t) \exp (j2\pi f_c t)]$$ (35)

$$s[l/W] = \Re [d[l] \exp (j2\pi f_c (l/W))] = \Re [d[l] \exp (j2\pi \zeta l)]$$ (36)

where $\zeta = f_c T$. 

PAPR of OFDM Systems

- In (36) \([d[0], \cdots, d[N_c - 1]]\) denotes the \(N_c\) point IDFT of the vector with the \(i^{th}\) component equal to

\[
\begin{cases} 
\tilde{d}_i & \text{when } i = lN_c/n \text{ for integer } l. \\
0 & \text{otherwise}
\end{cases}
\]  

(37)

- From (36), we can conclude that the maximum amplitude of the sequence \(\{s[l/W]\}\) depends on the amplitude of \(\{d[l]\}\).

- By definition

\[
d[i] = \frac{1}{\sqrt{N_c}} \sum_{k=0}^{n-1} \tilde{d}[kN_c/n] e^{\frac{j2\pi}{N_c} \frac{ikN_c}{n}}
\]  

(38)

- **Claim:** If we assume that the symbols \(\tilde{d}_i\) are chosen uniformly from a unit circle, then their distribution is invariant to rotations in the complex plane. Therefore their distribution is *circularly symmetric*. 
Note: A complex random vector \( \mathbf{X} \) is said to be Circularly symmetric (or Proper) if \( \mathbf{X} \sim e^{j\theta} \mathbf{X} \) for any \( \theta \).

- From the above definition for circular symmetry of a random vector, we can conclude that for \( \mathbb{E}[\mathbf{X}] = e^{j\theta} \mathbb{E}[\mathbf{X}] \) to be possible, \( \mathbb{E}[\mathbf{X}] = 0 \).
- Using the above conclusion we can verify the claim made in the previous slide. For example, let us assume that \( \tilde{d}_i \) are 8-ary PSK symbols i.e., \( \tilde{d}_i \in \left\{ e^{j\pi m/4}, m = 0, 1, \ldots, 7 \right\} \). Now suppose this points are chosen uniformly with probability \( 1/8 \), then \( \mathbb{E}[\tilde{d}_i] = 0 \), which verifies the above condition for circular symmetry of \( \tilde{d}_i \).
- (We will discuss in detail about Circularly Symmetric Random Vectors later in the course.)

- Also multiplication of \( \tilde{d}[kN_c/n] \) by the complex exponential \( e^{j2\pi ik/n} \) does not change the distribution of \( \tilde{d}[kN_c/n] \).
Therefore each term of (38) is identically distributed and circularly symmetric. Hence, we can invoke a version of the central limit theorem (CLT) for circularly symmetric random variables.

If we allow \( N_c \to \infty \) for a fixed \( n \) then the exponential term in (38) becomes undefined. Similarly, allowing \( n \to \infty \) for a fixed \( N_c \) is not possible since \( n \leq N_c \).

However a reasonable thing to do in such a case is let \( n, N_c \to \infty \) such that \( n/N_c \) is equal to \( \alpha \), a constant. For this case, using the CLT for circularly symmetric random variables, the quantity \( d[i] \) converges to a circularly symmetric complex Gaussian distribution with zero mean and variance \( \alpha \). To see this, we express \( d[i] \) in (38) as

\[
d[i] = \sqrt{\frac{n}{N_c}} \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \tilde{d}[kN_c/n] e^{\frac{j2\pi ikN_c}{n}}
\]

(39)

\( \overset{D}{\to} \mathcal{CN}(0,1) \) since \( \mathbb{E}[|\tilde{d}_i|^2] = 1 \)
PAPR of OFDM Systems (ctd...)

- From (39), since the underbraced term converges in distribution to $\mathcal{CN}(0, 1)$, therefore $d[i]$ (which is equal to $\alpha$ times the underbraced term in (39)) would converge to $\mathcal{CN}(0, \alpha)$.
- Therefore $|d[i]|$ is Rayleigh distributed with parameter $\alpha$. Since Rayleigh distribution has infinite support, therefore the peak value of the signal will exceed any given value with non-zero probability.
- Now since $|d[i]|$ is a r.v., therefore $|d[i]|^2$ is also a r.v., hence the PAPR defined in (24) is also a r.v. Therefore, we need to compute its cumulative distribution function (CDF) to analyze its behaviour.
- The CDF of PAPR $F_{\text{PAPR}}(x)$ is defined as

$$F_{\text{PAPR}}(x) = P(\text{PAPR} \leq x) = P(\max_l \frac{|d[l]|^2}{P_{\text{av}}} \leq x)$$

$$= P \left( \frac{|d[0]|^2}{P_{\text{av}}} \leq x, \cdots, \frac{|d[n]|^2}{P_{\text{av}}} \leq x \right)$$

$$= \left[ P \left( \frac{|d[0]|^2}{P_{\text{av}}} \leq x \right) \right]^n \quad (40)$$
PAPR of OFDM Systems (ctd...)

- Since $P_{av} = \alpha/2$, therefore we can re-write (40) as

$$F_{PAPR}(x) = \left[ P \left( \frac{|d[0]|^2}{\alpha} \leq \frac{x}{2} \right) \right]^n$$

$$= \left( 1 - e^{-x/2} \right)^n \quad (41)$$

where (41) follows by noting that $\frac{|d[0]|^2}{\alpha} \sim \exp(1)$ where $\exp(\lambda)$ denotes an exponential distribution with parameter $\lambda$.

- Now if we assume that at least 95% of the symbols shouldn’t get clipped, then the power amplifier bias setting $P_{bias}$ is chosen such that $P_{bias} \geq x$. The value of $x$ can be computed by solving the following transcendental equation. Here we assume number of subcarriers $n = N_c = 64$

$$\left( 1 - e^{-x/2} \right)^{64} = 0.95 \quad (42)$$

- Obviously, if $x$ is large then so is $P_{bias}$. This implies an higher rating power amplifier (PA), which means an increase in both cost and size of PA.
PAPR of OFDM Systems (ctd...)

- From (41), it is hard to understand the effect of number of subcarriers $N_c$ on PAPR. To understand the impact of $N_c$ on PAPR, we consider the complementary CDF of PAPR,

$$
P(\text{PAPR} > x) = P\left( \max_l \frac{|d[l]|^2}{P_{av}} > x \right)
\leq \sum_{l=0}^{n-1} P\left( \frac{|d[l]|^2}{P_{av}} > x \right)
= \sum_{l=0}^{n-1} P\left( \frac{|d[l]|^2}{\alpha} > \frac{x}{2} \right)
= ne^{-x/2}
$$

where (43) follows from the union bound i.e.,

$$P(\bigcup_i A_i) \leq \sum_i P(A_i),$$

(44) follows because $P_{av}/\alpha = 1/2$ and (45) follows by noting that $|d[l]|^2/\alpha \sim \exp(1)$.

- From (45), we notice that the upperbound on PAPR increases linearly with $n$. 

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Histogram of PAPR Random Variable with Number of Subcarriers $N_c = 64$

Histogram of PAPR Random Variable with Number of Subcarriers $N_c = 2048$
Therefore it is desirable to keep PAPR of the transmit signal low. Indeed, the high PAPR of an OFDM transmit signal is proving to be a major obstacle in using OFDM in the uplink of a wireless system.

Typically if a high PAPR signal is transmitted then it causes:

- Spectral regrowth.
- In-band distortion.

The nonlinear behaviour of a PA can be characterized by amplitude modulation/amplitude modulation (AM/AM) and amplitude modulation/phase modulation (AM/PM) responses.
PAPR of OFDM Systems (ctd...) 

- The typical AM/AM response of a PA is shown below:

- To avoid undesirable effects such as spectral regrowth and in-band distortion we
  - Transmit the high speak signal in the linear region by reducing the average power of the signal.
  - This is called *input backoff (IBO)*, which in turn results in *output backoff (OBO)*. This are depicted in the figure above.
  - This reduces the power efficiency of PA.
Inter-carrier Interference (ICI) in OFDM Systems

- The success of OFDM system depends on the orthogonality of sub-carriers.
  - For the subcarriers are to be orthogonal, the separation $\Delta f$ between the subcarriers must be an integer multiple of the signal bandwidth i.e., $\Delta f = k/T$, where $k$ is an integer and $T$ is OFDM symbol time.
- However in practice $\Delta f \neq k/T$ because of
  - Mismatched oscillators.
  - Doppler shifts.
  - Timing synchronization errors.
- For example if an oscillator is accurate to 0.1 parts per million (ppm) , then the frequency offset $f_{\text{offset}} \approx f_c(0.1 \text{ ppm})$. If $f_c = 5\text{GHz}$ (as in IEEE 802.11a WLAN systems), then $f_{\text{offset}} = 500 \text{ Hz}$.
- Even this small value of $f_{\text{offset}}$ would destroy the orthogonality of subcarriers, thereby causing ICI.
- In the following we will analyze the effect of ICI on OFDM systems.
ICI in OFDM Systems (ctd...)

- The (matched filter based) receiver output corresponding to the $l^{th}$ subcarrier can be written as

$$x_l(t) = e^{\frac{j2\pi lt}{T}}$$  

(46)

- The interfering sub-carrier $m$ can be written as

$$x_{l+m}(t) = e^{\frac{j2\pi (l+m)t}{T}}$$  

(47)

- Now if the sub-carrier in (47) is demodulated with a (fractional) frequency offset $\delta/T$ at the receiver, then the output is

$$\hat{x}_{l+m}(t) = e^{\frac{j2\pi (l+m+\delta)t}{T}}$$  

(48)
ICI in OFDM Systems (ctd...)

- The ICI $I_m$ between the sub-channels due to the frequency mismatch is given by the inner product between $x_l$ and $\hat{x}_{l+m}$

$$I_m = \int_0^T x_l(t)\hat{x}_{l+m}^*(t)dt = \frac{T(1 - e^{-j2\pi(\delta+m)})}{j2\pi(\delta + m)}$$ (49)

- As can be seen from (49) $\delta = 0 \implies I_m = 0$ as expected.
- The total ICI power on sub-carrier $i$ due to all interfering sub-carriers can be written as

$$\text{ICI}_i = \sum_{m\neq i} |I_m|^2 = T^2 \sum_{m\neq i} \text{sinc}^2(m + \delta)$$

$$\approx C_0 (T\delta)^2,$$ (50)

where $C_0$ is a constant.
ICl in OFDM Systems (ctd...)  

- From (50), the following conclusions can be drawn:
  - The ICI power increases quadratically with respect to the frequency offset $\delta$.
  - Also as the OFDM symbol time $T$ increases the ICI also increases. This is because as $T$ increases the sub-carriers tend to get narrow and hence more closely spaced, which then results in greater ICI.
  - A quick look at (50) suggests that ICI does not depend directly on the number of sub-carriers $N_c$. However if $N_c$ increases then so does $T$, whose increase in turn increases ICI as explained above.

- From this discussion and the preceding discussion on PAPR we can conclude the following:
  - By picking $N_c$ to be large one can improve the spectral efficiency of an OFDM system by minimizing the loss due to cyclic prefix.
  - However an increase in $N_c$ increases both PAPR and ICI of the OFDM system.
  - Therefore $N_c$ should be chosen judiciously so that the spectral efficiency, PAPR and ICI conditions are met.