Dhirubhai Ambani Institute of Information and Communication Technology (DA-IICT)
Mid-semester Examination
CT314 (Statistical Communication Theory)
Date of Examination: March 24, 2017
Duration: 2 Hours
Maximum Marks: 25

Instructions:
1. Attempt all questions.
2. Use of scientific non programmable calculator is permitted.
3. Figures in brackets indicate full marks.
4. All the acronyms carry their usual meaning.

Q1: Let X and Y be two random variables with \( Y = cX + d \), where \( c \), \( d \) are constants. Find the correlation coefficient between X and Y. (2)

Q2: Consider a vector of random variables \( X = [X_1, X_2]^T \). These random variables have unit variance and are uncorrelated. Now the transformed vector \( Y = AX \), where \( A \) is the transformation matrix. Find the matrix \( A \) so that \( Y \) has the covariance matrix \( C_Y = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \) (8)

Q3: Let \( X_1 \), \( X_2 \), and \( X_3 \) be the zero mean random variables having same variance. We wish to predict \( X_3 \) as \( aX_1 + bX_2 \), \( a \) and \( b \) are constants. (a) Find the MMSE estimate of \( X_3 \). Now assuming that covariance does not depend on the specific index of random variables, but rather on the distance between them [meaning \( COV(X_1, X_2) = COV(X_2, X_3) \)], express \( a \) and \( b \) in terms of correlation coefficients. (7)

Q4: Consider jointly Gaussian random variables \( X_1 \) and \( X_2 \) with mean vector \( m_x \) and covariance matrix \( C_X \). Now define \( Y = AX \) to get \( Y_1 \) and \( Y_2 \), where \( A \) is a invertible matrix. (a) Show that \( Y \) is jointly Gaussian (b) Write the mean vector and covariance matrix for the vector \( Y \). (c) Now choose \( A \) to make \( Y \) as statistically independent (d) Reason out why \( A \) has to be invertible. (8)

"BEST WISHES"
1. \( y = cx + d \)

\[ \text{Cov}(x,y) = E\left[ (x-m_x)(y-m_y) \right] \]

\( y = cx + d \) so \( m_y = cE(x) + d = cm_x + d \)

\( y - m_y = c(x - m_x) \)

\[ \text{Cov}(x,y) = E\left[ (x-m_x)(y-m_y) \right] = E \left[ c(x-m_x)^2 \right] = c \sigma_x^2 \]

\[ \sigma_y^2 = E(y - m_y)^2 = c^2 \sigma_x^2 \]

\[ p_{xy} = \frac{c \sigma_x^2}{\sigma_x \sigma_x} = 1 \]

2. \( C_y = \begin{bmatrix} 1 & 1.5 \\ 1.5 & 1 \end{bmatrix} \)

Eigen vectors \( U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

\( \Sigma = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \)

\( A = U \Sigma U^T \)

Verify \( C_y = A A^T \)

3. \( x_3 = a x_1 + b x_2 \)

Minimize \( E \left( x_3 - \hat{x}_3 \right)^2 \)

Differentiate \( a \) and \( b \) and equate to 0.
we set
\[
\begin{bmatrix}
E(X_1) & E(X_1 X_2) \\
E(X_1 X_2) & E(X_2)
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix} = \frac{\begin{bmatrix}
E(X_1 X_3) \\
E(X_2 X_3)
\end{bmatrix}}{egin{bmatrix}
\sigma_{X_1}^2 \\
\sigma_{X_2}^2
\end{bmatrix}}
\]

So solving these two equations,
\[
a = \sqrt{\frac{\sigma_{X_1}^2}{\sigma_{X_2}^2}} \left( \text{Cov}(X_1, X_3) - \text{Cov}(X_1 X_2) \text{Cov}(X_2, X_3) \right)
\]
\[
b = \frac{\sigma_{X_1}^2}{\sigma_{X_2}^2} \left( \text{Cov}(X_2, X_3) - \text{Cov}(X_1 X_2) \text{Cov}(X_1, X_3) \right)
\]

New given:
\[
\text{Cov}(X_1 X_2) = \text{Cov}(X_1 X_3)
\]
\[
p_1 \sigma_{X_1}^2 = p_2 \sigma_{X_2}^2
\]

\[
a = \frac{\sigma_{X_1}^2}{\sigma_{X_2}^2} \left( p_2 - p_1 \right)
\]
\[
b = \frac{\sigma_{X_1}^2}{\sigma_{X_2}^2} \left( 1 - p_1^2 \right)
\]

So verify
\[
a = \frac{p_2 - p_1}{1 - p_1^2}
\]
\[
b = \frac{p_1(1 - p_2)}{1 - p_1^2}
\]
\[ Y = AX \]

\[
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[ f_{y_1, y_2}(y_1, y_2) = \frac{f_{x_1, x_2}(x_1, x_2)}{|J(\frac{y_1, y_2}{x_1, x_2})|} \]

\[ |J| = \begin{vmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2}
\end{vmatrix} = \begin{vmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{vmatrix} = \mathbf{det}(A)
\]

\[ f_{x_1, x_2}(x_1, x_2) = -\frac{1}{2} (x - m_x)^T C_x^{-1} (x - m_x) \]

\[ f_{x_1 x_2}(x_1, x_2) = \frac{1}{2\pi} \mathbf{det}(A) \frac{1}{\sqrt{C_x}} \]

\[ AX = Y \]

\[ N \cdot (A^{-1} y - m_x) = A^{-1} (y - A m_x) = (A^{-1} y - m_x)^T A^{-1} (y - A m_x) = \frac{1}{2} (y - m_y)^T (A^{-1})^T C_y A^{-1} (y - m_y) \]

\[ e^{-\frac{1}{2} (y - m_y)^T C_y^{-1} (y - m_y)} \]

\[ (2\pi)^{-\frac{1}{2}} \mathbf{det}(C_y)^{-\frac{1}{2}} \]

Covariance matrix \( Y \)

\[ C_y = (A C_x A^T)^{-1} \]

\[ \det(C_y) = \mathbf{det}(A)^{-2} \mathbf{det}(C_x) \]

\[ (AC_y A^T)^{-1} \]

\[ e^{-\frac{1}{2} (y - m_y)^T C_y^{-1} (y - m_y)} \]

\[ (2\pi)^{-\frac{1}{2}} \mathbf{det}(C_y)^{-\frac{1}{2}} \]

\[ \text{represents joint pdf of } y_1 \text{ and } y_2 \]

\[ \text{which is jointly Gaussian pdf} \]
For $y_1$ and $y_2$ to be independent, they have to be uncorrelated for Gaussian case.

So \[ y = A x \]

... A is chosen as

\[ A \perp U \text{ i.e., eigenvector matrix} \]

\[ C_x \] (Covariance matrix $y_x$)

Then $y_1$ and $y_2$ are uncorrelated, hence independent.

(c) $y A$ is not invertible, determinant $f^j$ Jacobian matrix becomes 0. So joint pdf cannot be determined, i.e., $f_{y_1, y_2}$