Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(1) \( \left( z - 3 \right) \sin \frac{\pi}{z + 2} \); \( z = -2 \)

(2) \( \frac{z - 5}{z^3} \); \( z = 0 \)

(3) \( \frac{z}{(z+1)(z+2)} \); \( z = -2 \)

(4) \( \frac{1}{z^2 (z-3)^2} \); \( z = 3 \)

Let \( z + 2 = u \rightarrow z = u - 2 \)

\( (z - 3) \sin \frac{1}{u + 2} \) = \( (u - 5) \sin \frac{1}{u} \)

\( = \left( u - 5 \right) \left\{ \frac{1}{u} - \frac{1}{2! u^2} + \frac{5}{6} u \right\} \)

\( = 1 - \frac{5}{u} - \frac{1}{2! u^2} + \frac{5}{6} u \)

\( = \left( 1 - \frac{5}{2+2} \right) - \frac{1}{6 (2+2)^2} + \frac{5}{6 (2+2)^3} + \frac{1}{120 \left( 2+2 \right)^5} \)

\( z = -2 \) is an essential singularity.

The series converges for all values of \( z \neq -2 \).
(b)  \[ \frac{x - 5u^2}{x^3} = \frac{1}{2^3} \left\{ x - \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right) \right\} \]

\[ = \frac{1}{2^3} \left\{ \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \right\} \]

\[ = \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} + \cdots \]

\( z = 0 \) is a removable singularity.

The series converges for all values of \( z \).

(c) Let \( z + 2 = u \).

\[ \frac{x}{(z+1)(z+2)} = \frac{u^2}{(u-1)u} = \frac{u-1}{u} \cdot \frac{1}{1-u} \]

\[ = \frac{1}{u} \left\{ 1 + u + u^2 + u^3 + \cdots \right\} \]

\[ = \frac{1}{u} + 1 + u^2 + \cdots \]

\[ = \frac{2}{u+1} + (u+1) + (u+2)^2 + \cdots \]

\( z = -2 \) is a pole of order 1, or simple pole.

The series converges for all values of \( z \) s.t.

\[ 0 < |z+1| < 1 \]

(d) Let \( z - 3 = u \). Then by the binomial thm.,

\[ \frac{1}{2^1(2-3)^2} = \frac{1}{u^2(u+3)^2} = \frac{1}{9u^2(1+\frac{u}{3})^2} \]

\[ = \frac{1}{9u^2} \left\{ 1 + \left( -2 \cdot \frac{u}{3} \right) + \frac{(-2)(-3)}{2!} \left( \frac{u}{3} \right)^2 + \frac{(-2)(-3)(-4)}{3!} \left( \frac{u}{3} \right)^3 + \cdots \right\} \]

\[ = \frac{1}{9u^2} - \frac{2}{27} u + \frac{1}{243} u^2 + \cdots \]

\[ = \frac{1}{9(2-3)^2} - \frac{2}{27} (z-3) + \frac{1}{243} (z-3)^2 + \cdots \]

Q. 1 Expand \( f(z) = \frac{1}{(z+1)(z+3)} \) in a Laurent series valid

for (a) \( 1 < 1 + 1 < 3 \) (b) \( 1 + 1 > 3 \); (c) \( 0 < 1 + 1 < 2 \)

d) \( 1 + 1 < 1 \).
\[ f(2) = \frac{1}{(2+1)(2+3)} = \frac{1}{6} \left[ \frac{1}{(2+1)} - \frac{1}{(2+3)} \right] \]

If \( 121 > 1 \)

\[
\frac{1}{2(2+1)} = \frac{1}{2} \left( 1 - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^3} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \ldots
\]

If \( 121 < 3 \)

\[
\frac{1}{4(2+3)} = \frac{1}{6} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots \right)
\]

\[
= \frac{1}{6} - \frac{1}{18} + \frac{1}{36} - \frac{1}{18} + \ldots
\]

Then the required Laurent expansion valid for both \( 121 > 1 \) and \( 121 < 3 \) i.e., \( 1 < 121 < 3 \), is

\[
\ldots - \frac{1}{2^4} + \frac{1}{2^3} - \frac{1}{2^2} + \frac{1}{2} - \frac{1}{18} + \frac{1}{36} - \frac{1}{18} + \ldots
\]

If \( 121 > 3 \)

\[
\frac{1}{2(2+3)} = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots \right)
\]

The required Laurent expansion for both \( 121 > 1 \) and \( 121 > 3 \), i.e., \( 121 > 3 \), is by substitution

\[
\frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \ldots
\]

If \( 0 < 121 < 2 \)

\[
\frac{1}{2+1} = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \ldots
\]

If \( 121 < 1 \)

\[
\frac{1}{2+1} = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \ldots
\]

If \( 121 < 3 \) we have in part \( C \)

\[
\frac{1}{2+1} = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \ldots
\]

\[ \text{If } 121 < 3 \text{ we have in part } C. \]

\[
\frac{1}{2+1} = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \ldots
\]

\[ \text{If } 121 < 3 \text{ we have in part } C. \]

\[
\frac{1}{2+1} = \frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} + \ldots
\]

\[ \text{If } 121 < 3 \text{ we have in part } C. \]
Show that \[ \int_{0}^{\theta} \frac{\cos 3\theta + \theta}{5 - 4 \cos \theta} \, d\theta = \frac{\pi}{12}. \]

\[ \cos \theta = \frac{2 + e^{-i\theta}}{2}, \quad \cos 3\theta = \frac{2^3 + e^{-3i\theta}}{2} = \frac{2^3 + e^{-3\theta}}{2}, \]

\[ 2 - e^{-i\theta} \quad d\theta = i2 + \theta \]

\[ \int_{0}^{\theta} \frac{\cos 3\theta + \theta}{5 - 4 \cos \theta} \, d\theta = \int_{c} \frac{(2^3 + e^{-3\theta})}{\pi (5 - (2 + e^{-\theta}) (2 - 2))} \, d\theta \]

\[ = -\frac{1}{2} \int_{c} \frac{2e^{\theta} + 1}{\pi (2(2 - 1)(2 - 2))} \, d\theta \]

Poles with \( c \) are \( z = 0 \) of order 2 and \( z = \frac{1}{2} \).

Residue at \( z = 0 \) is:

\[ \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left[ \frac{2e^{\theta} + 1}{2^2 (2 - 1)(2 - 2)} \right] = \frac{1}{8} \]

Residue at \( z = \frac{1}{2} \) is:

\[ \lim_{z \to \frac{1}{2}} \left[ (2 - \frac{1}{2}) \frac{2e^{\theta} + 1}{2^2 (2 - 1)(2 - 2)} \right] = -\frac{65}{24} \]

Then:

\[ -\frac{1}{\pi} \int_{c} \frac{2e^{\theta} + 1}{2^2 (2 - 1)(2 - 2)} \, d\theta = -\frac{1}{\pi} \left( \frac{21}{65} \right) \left[ \frac{\pi}{6} - \frac{65}{24} \right] \]

\[ = \frac{\pi}{12}. \]
Solved Problems

RESIDUES AND THE RESIDUE THEOREM

1. Let \( f(z) \) be analytic inside and on a simple closed curve \( C \) except at point \( a \) inside \( C \).

\( (a) \) Prove that

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n
\]

where

\[
a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz, \quad n = 0, \pm 1, \pm 2, \ldots
\]

i.e., \( f(z) \) can be expanded into a converging Laurent series about \( z = a \).

\( (b) \) Prove that

\[
\oint_C f(z) \, dz = 2\pi i a_{-1}
\]

\( (a) \) This follows from Problem 25 of Chapter 6.

\( (b) \) If we let \( \kappa = -1 \) in the result of \( (a) \), we find

\[
a_{-1} = \frac{1}{2\pi i} \oint_C f(z) \, dz, \quad \text{i.e.} \quad \oint_C f(z) \, dz = 2\pi i a_{-1}
\]

We call \( a_{-1} \) the residue of \( f(z) \) at \( z = a \).

(2) Prove the residue theorem. If \( f(z) \) is analytic inside and on a simple closed curve \( C \) except at a finite number of points \( a, b, c, \ldots \) inside \( C \) at which the residues are \( a_{-1}, b_{-1}, c_{-1}, \ldots \) respectively, then

\[
\oint_C f(z) \, dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots)
\]

i.e., \( 2\pi i \) times the sum of the residues at all singularities enclosed by \( C \).

With centres at \( a, b, c, \ldots \) respectively construct circles \( C_a, C_b, C_c, \ldots \) which lie entirely inside \( C \) as shown in Fig. 7-4. This can be done since \( a, b, c, \ldots \) are interior points. By Theorem 5, Page 97, we have

\[
\oint_C f(z) \, dz = \oint_{C_a} f(z) \, dz + \oint_{C_b} f(z) \, dz + \oint_{C_c} f(z) \, dz + \cdots
\]

But by Problem 1,

\[
\oint_{C_a} f(z) \, dz = 2\pi i a_{-1}, \quad \oint_{C_b} f(z) \, dz = 2\pi i b_{-1}, \quad \oint_{C_c} f(z) \, dz = 2\pi i c_{-1}, \ldots
\]

Then from (1) and (2) we have, as required,

\[
\oint_C f(z) \, dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) = 2\pi i (\text{sum of residues})
\]

The proof given here establishes the residue theorem for simply-connected regions containing a finite number of singularities of \( f(z) \). It can be extended to regions with infinitely many isolated singularities and to multiply-connected regions (see Problems 96 and 91).

3. Let \( f(z) \) be analytic inside and on a simple closed curve \( C \) except at a pole \( a \) of order \( m \) inside \( C \). Prove that the residue of \( f(z) \) at \( a \) is given by

\[
a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}((z-a)^m f(z))
\]

Method 1. If \( f(z) \) has a pole \( a \) of order \( m \), then the Laurent series of \( f(z) \) is

\[
f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots
\]
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Then multiplying both sides by \((z-a)^m\), we have

\[
(x-a)^m f(x) = a_{-m} + a_{-m+1}(x-a) + \cdots + a_{-1}(x-a)^{m-1} + a_0(x-a)^m + \cdots \quad (8)
\]

This represents the Taylor series about \(z=a\) of the analytic function on the left. Differentiating both sides \(m-1\) times with respect to \(z\), we have

\[
\frac{d^{m-1}}{dz^{m-1}} \left[ (z-a)^m f(z) \right] = (m-1)! a_{-1} + m(m-1) \cdots 2a_0 (x-a) + \cdots
\]

Thus on letting \(z \to a\),

\[
\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[ (z-a)^m f(z) \right] = (m-1)! a_{-1}
\]

from which the required result follows.

**Method 1.** The required result also follows directly from Taylor's theorem on noting that the coefficient of \((z-a)^{m-1}\) in the expansion (8) is

\[
a_{-1} = \frac{1}{(m-1)!} \left. \frac{d^{m-1}}{dz^{m-1}} \left[ (z-a)^m f(z) \right] \right|_{z=a}
\]

**Method 2.** See Problem 28, Chapter 5, Page 132.

4. Find the residues of
   (a) \(f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2 + 4)}\)
   (b) \(f(z) = e^z \csc^2 z\) at all its poles in the finite plane.

(a) \(f(z)\) has a double pole at \(z = -1\) and simple poles at \(z = \pm 2i\).

**Method 1.**

\[
\text{Residue at } z = -1 \text{ is}
\]

\[
\lim_{z \to -1} \frac{d}{dz} \left[ \left( z + 1 \right)^2 \frac{z^2 - 2z}{(z+1)^2(z^2 + 4)} \right] = \lim_{z \to -1} \frac{(z+4)(2z-2) - (z^2 - 2z)(2z)}{(z^2 + 4)^2} = -\frac{14}{25}
\]

Residue at \(z = 2i\) is

\[
\lim_{z \to 2i} \left( z - 2i \right)^2 \frac{z^2 - 2z}{(z+1)^2(z^2 + 4)} = \frac{-4 - 4i}{(2i+1)^2} \frac{1}{(2i+1)^2} = \frac{7 + i}{25}
\]

Residue at \(z = -2i\) is

\[
\lim_{z \to -2i} \left( z + 2i \right)^2 \frac{z^2 - 2z}{(z+1)^2(z^2 + 4)} = \frac{-4 + 4i}{(-2i+1)^2} \frac{1}{(-2i+1)^2} = \frac{7 - i}{25}
\]

**Method 2.**

Residue at \(z = 2i\) is

\[
\lim_{z \to 2i} \left( z - 2i \right)^2 \frac{z^2 - 2z}{(z+1)^2(z^2 + 4)} = \left\{ \lim_{z \to 2i} \frac{z^2 - 2z}{(z+1)^2} \right\} \left\{ \lim_{z \to 2i} \frac{z - 2i}{z^2 + 4} \right\} = \frac{-4 - 4i}{(2i+1)^2} \frac{1}{(2i+1)^2} = \frac{7 - i}{25}
\]

taking an isolated order \(m\) pole.

(b) \(f(z) = e^z \csc^2 z\) has double poles at \(z = 0, \pm \pi, \pm 2\pi, \ldots\), i.e. \(z = m\pi\) where \(m = 0, \pm 1, \pm 2, \ldots\).

**Method 1.**

Residue at \(z = m\pi\) is

\[
\lim_{z \to m\pi} \frac{d}{dz} \left[ \left( z - m\pi \right)^2 \frac{e^z}{\sin^2 z} \right] = \lim_{z \to m\pi} \frac{e^z (z - m\pi)^2 \sin z + 2(z - m\pi) \sin z - 2(z - m\pi)^2 \cos z}{\sin^3 z}
\]

\[
= \lim_{z \to m\pi} \frac{e^z (z - m\pi)^2 \sin z + 2(z - m\pi) \sin z - 2(z - m\pi)^2 \cos z}{\sin^3 z}
\]
THE RESIDUE THEOREM. EVALUATION OF INTEGRALS AND SERIES

Letting \( z = m\pi = u + m\pi \), this limit can be written

\[
\lim_{u \to \pi} \frac{u^3 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u}
\]

\[= e^{m\pi} \lim_{u \to 0} \frac{u^3 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u}
\]

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that

\[
\lim_{u \to 0} \frac{u^3}{\sin^3 u} = 1
\]

and thus write the limit as

\[
e^{m\pi} \lim_{u \to 0} \frac{u^3 \sin u + 2u \sin u - 2u^2 \cos u}{u^3}
\]

which is obtained using L'Hospital's rule several times. In evaluating this limit we can instead use the series expansions \( \sin u = u - u^3/6 + \cdots \), \( \cos u = 1 - u^2/2! + \cdots \).

**Method 2** (using Laurent's series).

In this method we expand \( f(z) = e^z \csc^2 z \) in a Laurent series about \( z = m\pi \) and obtain the coefficient of \( 1/(z - m\pi) \) as the required residue. To make the calculation easier let \( z = u + m\pi \).

Then the function to be expanded in a Laurent series about \( u = 0 \) is \( e^{m\pi + u} \csc^2 (m\pi + u) = e^{m\pi} e^u \csc^2 u \). Using the Maclaurin expansions for \( e^u \) and \( \sin u \), we find using long division

\[
e^{m\pi} e^u \csc^2 u = 
\frac{e^{m\pi} \left( 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \right)}{u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots} = \frac{e^{m\pi} \left( 1 + u + \frac{u^2}{2!} + \cdots \right)}{u^2 \left( 1 - \frac{u^2}{6} + \frac{u^4}{120} - \cdots \right)}
\]

and so the residue is \( e^{m\pi} \).

5. Find the residue of \( F(z) = \frac{\cot z \coth z}{z^2} \) at \( z = 0 \).

We have as in Method 2 of Problem 4(b),

\[
F(z) = \frac{\cos z \coth z}{z^{2} \sin z \sinh z} = \frac{\left( 1 - \frac{z^4}{2!} + \frac{z^6}{4!} - \cdots \right) \left( 1 + \frac{z^4}{2!} + \frac{z^6}{4!} + \cdots \right)}{z^2 \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) \left( z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \right)} = \frac{\left( 1 - \frac{z^4}{6} + \cdots \right)}{z^3 \left( 1 - \frac{z^4}{30} + \cdots \right)}
\]

and so the residue (coefficient of \( 1/z \)) is \( -7/46 \).

**Another method.** The result can also be obtained by finding

\[
\lim_{z \to 0} \frac{1}{2i} \int_{C} \frac{e^{z}}{z^2 (z^2 + 2z + 2)} dz
\]

but this method is much more laborious than that given above.

6. Evaluate \( \frac{1}{2\pi i} \int_{C} \frac{e^{z}}{z^2 (z^2 + 2z + 2)} dz \) around the circle \( C \) with equation \( |z| = 3 \).

The integrand \( e^{z}/(z^2 + 2z + 2) \) has a double pole at \( z = 0 \) and two simple poles at \( z = -1 \pm i \) (roots of \( z^2 + 2z + 2 = 0 \)). All these poles are inside \( C \).
Residue at \( z = 0 \) is

\[
\lim_{z \to 0} \frac{1}{z - 0} \frac{e^z}{z^2(z^2 + 2z + 2)} = \lim_{z \to 0} \frac{(z^2 + 2z + 2)(e^z) - (e^z)(2z + 2)}{(z^2 + 2z + 2)^2} = \frac{t}{2}
\]

Residue at \( z = -1 + i \) is

\[
\lim_{z \to -1 + i} \left( z - (-1 + i) \right) \frac{e^z}{z^2(z^2 + 2z + 2)} = \lim_{z \to -1 + i} \left( \frac{e^z}{z} \right) \lim_{z \to -1 + i} \left( \frac{z + 1 - i}{z^2 + 2z + 2} \right) = \frac{e^{(-1 + i)t}}{(-1 + i)^2} \cdot \frac{1}{2t} = \frac{e^{(-1 + i)t}}{4}
\]

Residue at \( z = -1 - i \) is

\[
\lim_{z \to -1 - i} \left( z - (-1 - i) \right) \frac{e^z}{z^2(z^2 + 2z + 2)} = \frac{e^{(-1 - it)}}{4}
\]

Then by the residue theorem

\[
\oint_C \frac{e^z}{z^2(z^2 + 2z + 2)} \, dz = 2\pi i \left( \text{sum of residues} \right)
\]

\[
= 2\pi i \left( \frac{t}{2} + \frac{e^{-1 + it}}{4} + \frac{e^{-1 - it}}{4} \right)
\]

i.e.,

\[
\frac{1}{2\pi i} \oint_C \frac{e^z}{z^2(z^2 + 2z + 2)} \, dz = \frac{t - 1}{2} + \frac{1}{2} e^{-t} \cos t
\]

DEFINITE INTEGRALS OF THE TYPE \( \int_{-R}^{R} F(x) \, dx \)

7. If \( |F(x)| \leq \frac{M}{R^k} \) for \( z = Re^{it} \) where \( k > 1 \) and \( M \) are constants, prove that \( \lim_{R \to \infty} \int_{-R}^{R} F(x) \, dx = 0 \) where \( \Gamma \) is the semicircular arc of radius \( R \) shown in Fig. 7.5.

By Property 5, Page 93, we have

\[
\left| \int_{\Gamma} F(z) \, dz \right| \leq \frac{M}{R^k} \cdot 2R = \frac{2M}{R^{k-1}}
\]

since the length of arc \( L = 2R \). Then

\[
\lim_{R \to \infty} \int_{\Gamma} F(z) \, dz = 0 \quad \text{and so} \quad \lim_{R \to \infty} \int_{-R}^{R} F(x) \, dx = 0
\]

8. Show that for \( z = Re^{it} \), \( |f(z)| \leq \frac{M}{R^k} \), \( k > 1 \) if \( f(z) = \frac{1}{z^2 + 1} \).

If \( z = Re^{it} \), \( |f(z)| = \left| \frac{1}{Re^{it} + 1} \right| \leq \left| \frac{1}{Re^{it}} - 1 \right| = \frac{1}{R^2 - 1} \leq \frac{2}{R^2} \) if \( R \) is large enough (say \( R > 2 \), for example) so that \( M = 2. k = 6 \).

Note that we have made use of the inequality \( |s_1 + s_2| \leq |s_1| + |s_2| \) with \( s_1 = Re^{it} \) and \( s_2 = 1 \).

9. Evaluate \( \int_{0}^{\infty} \frac{dx}{x^2 + 1} \).

Consider \( \oint_{C} \frac{dz}{z^2 + 1} \), where \( C \) is the closed contour of Fig. 7.5 consisting of the line from \(-R\) to \( R \) and the semicircle \( \gamma \), traversed in the positive (counterclockwise) sense.
THE RESIDUE THEOREM. EVALUATION OF INTEGRALS AND SERIES (CHAP. 7)

Since $z^2 + 1 = 0$ when $z = e^{in\theta}$, $e^{in\theta}$, $e^{i(n+1)\theta}$, $e^{(n+1)i\theta}$, $e^{(n+1)i\theta}$, these are simple poles of $1/(z^2 + 1)$. Only the poles $e^{in\theta}$ and $e^{i(n+1)\theta}$ lie within $C$. Then using L'Hopital's rule,

Residue at $e^{i(n+1)\theta} = \lim_{z \to e^{i(n+1)\theta}} \left( (z - e^{i(n+1)\theta}) \frac{1}{z^2 + 1} \right) = \lim_{z \to e^{i(n+1)\theta}} \frac{1}{2z} = \frac{1}{2} e^{-2i(n+1)\theta}$

Residue at $e^{i(n\theta)} = \lim_{z \to e^{i(n\theta)}} \left( (z - e^{i(n\theta)}) \frac{1}{z^2 + 1} \right) = \lim_{z \to e^{i(n\theta)}} \frac{1}{2z} = \frac{1}{2} e^{-2in\theta}$

Thus

$$\int_C \frac{dz}{z^2 + 1} = 2\pi i \left( \frac{1}{2} e^{-2i(n+1)\theta} + \frac{1}{2} e^{-2i(n\theta)} + \frac{1}{2} e^{-2i(n+1)\theta} \right) = \frac{2\pi}{3}$$

i.e.,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} + \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \frac{2\pi}{3} \quad (f)$$

Taking the limit of both sides of $(f)$ as $R \to \infty$ and using Problems 7 and 8, we have

$$\lim_{R \to \infty} \int_{R}^{\infty} \frac{dx}{x^2 + 1} = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{3} \quad (g)$$

Since $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{3}$, the required integral has the value $\pi/3$.

(10) Show that

$$\int_{-\infty}^{\infty} \frac{x^2 \sin x}{(x^2 + 1)^2 [x^2 + 2x + 2]} \, dx = \frac{7\pi}{50}$$

The poles of $\frac{x^2}{(x^2 + 1)^2 [x^2 + 2x + 2]}$ enclosed by the contour $C$ of Fig. 7-5 are $x = i$ of order 2 and $x = -1 + i$ of order 1.

Residue at $s = i$ is

$$\lim_{s \to i} \frac{ds}{I_{-1}^{\infty} \frac{1}{(s^2 + 1)^2 (x^2 + 2x + 2)}} = \frac{9i - 12}{100}$$

Residue at $s = -1 + i$ is

$$\lim_{s \to -1 + i} (s - (-1 - i)) \frac{x^2}{(s^2 + 1)^2 (x^2 + 2x + 2)} = \frac{3 - 4i}{25}$$

Then

$$\int_C \frac{x^2 \sin x \, dx}{(x^2 + 1)^2 [x^2 + 2x + 2]} = 2\pi i \left( \frac{9i - 12}{100} + \frac{3 - 4i}{25} \right) = \frac{7\pi}{50}$$

or

$$\int_{-\infty}^{\infty} \frac{x^2 \sin x \, dx}{(x^2 + 1)^2 [x^2 + 2x + 2]} + \int_{-\infty}^{\infty} \frac{x^2 \sin x \, dx}{(x^2 + 1)^2 [x^2 + 2x + 2]} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem 7, we obtain the required result.

DEFINITE INTEGRALS OF THE TYPE $\int_{0}^{2\pi} G(\sin \theta, \cos \theta) \, d\theta$

11. Evaluate $\int_{0}^{2\pi} \frac{d\theta}{3 - 2 \cos \theta + \sin \theta}$

Let $s = e^{i\theta}$. Then $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{s - s^{-1}}{2i}$, $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{s + s^{-1}}{2}$, $ds = is \, d\theta$ so that

$$\int_{0}^{2\pi} \frac{d\theta}{3 - 2 \cos \theta + \sin \theta} = \int_{0}^{2\pi} \frac{ds/2}{(s - 3(\cos \theta - \sin \theta)^{-1})/2i + (s - s^{-1})/2i} = \int_{0}^{2\pi} \frac{2 \, ds}{s (1 - 2i)(s + 6i - 1 - 2i)}$$

where $C$ is the circle of unit radius with centre at the origin (Fig. 7-6).
The poles of \( \frac{2}{(1 - 2i)x^2 + 6ix - 1 - 2i} \) are the simple poles

\[
s = \frac{-6i \pm \sqrt{(6i)^2 - 4(1 - 2i)(-1 - 2i)}}{2(1 - 2i)} = \frac{-6i \pm 4i}{2(1 - 2i)} = 2 - i, \quad (2 - 0)/5
\]

Only \((2 - 0)/5\) lies inside \(C\).

Residue at \((2 - 0)/5\)

\[
= \lim_{s \to (2 - 1)/5} \left(s - \frac{2}{2(1 - 2i)x + 6i} \right) \left(1 - 2i s^2 + 6i s - 1 - 2i \right)
\]

by L'Hospital's rule.

Then

\[
\oint_C \frac{2 \, ds}{(1 - 2i)z^2 + 6iz - 1 - 2i} = 2\pi i \left( \frac{1}{2i} \right) = \pi, \quad \text{the required value.}
\]

12. Show that \(\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \quad \text{if } a > |b|\).

Let \(s = e^{i\theta}\). Then \(\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{s - s^{-1}}{2i}\), \(ds = ie^{|\theta|}d\theta = is \, d\theta\) so that

\[
\int_0^{2\pi} \frac{ds}{a + b \sin \theta} = \oint_C \frac{ds/\theta}{a + b(s - s^{-1})/2i} = \oint_C \frac{2 \, ds}{bz^2 + 2aiz - b}
\]

where \(C\) is the circle of unit radius with centre at the origin, as shown in Fig. 7.6.

The poles of \(\frac{2}{bz^2 + 2aiz - b}\) are obtained by solving \(bz^2 + 2aiz - b = 0\) and are given by

\[
s = \frac{-a \pm \sqrt{a^2 - b^2}}{b} = \frac{-a \pm \sqrt{a^2 - b^2} i}{b}
\]

Only \(\frac{-a + \sqrt{a^2 - b^2}}{b} i\) lies inside \(C\), since

\[
\left| \frac{-a + \sqrt{a^2 - b^2}}{b} i \right| = \left| \frac{\sqrt{a^2 - b^2} - a}{b} \right| < 1 \quad \text{if } a > |b|
\]

Residue at \(s_1 = \frac{-a + \sqrt{a^2 - b^2} i}{b}\)

\[
= \lim_{s \to s_1} \frac{2}{bz^2 + 2aiz - b} = \lim_{s \to s_1} \frac{2}{bz^2 + 2aiz - b} = \frac{1}{b z_1 + ai} = \frac{1}{\sqrt{a^2 - b^2} i}
\]

by L'Hospital's rule.

Then

\[
\oint_C \frac{2 \, ds}{bz^2 + 2aiz - b} = 2\pi i \left( \frac{1}{\sqrt{a^2 - b^2} i} \right) = \frac{2\pi}{\sqrt{a^2 - b^2}}, \quad \text{the required value.}
\]

13. Show that \(\int_0^{2\pi} \frac{\cos \theta}{5 - 4 \cos \theta} \, d\theta = \frac{\pi}{12}\).

If \(s = e^{i\theta}\), then \(\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{s^2 + s^{-2}}{2}, \quad ds = is \, d\theta\) so that

\[
\int_0^{2\pi} \frac{\cos \theta}{5 - 4 \cos \theta} \, d\theta = \oint_C \frac{\left(\frac{s^2 + s^{-2}}{2}\right) \, ds}{5 - 4(s + s^{-1})/2 is} = \frac{1}{2i} \oint_C \frac{s^3 + 1}{s^2 + 1} \, ds
\]

where \(C\) is the contour of Fig. 7.6.

The integrand has a pole of order 3 at \(s = 0\) and a simple pole \(s = \frac{1}{2}\) inside \(C\).
THE RESIDUE THEOREM. EVALUATION OF INTEGRALS AND SERIES  [CHAP. 7

Residue at \( z = 0 \) is
\[
\lim_{z \to 0} \frac{d}{dz} \left( \frac{z^4 + 1}{z^5(2z - 1)(z - 2)} \right) = \frac{21}{8}.
\]

Residue at \( z = \frac{1}{2} \) is
\[
\lim_{z \to \frac{1}{2}} \frac{d}{dz} \left( \frac{z^4 + 1}{z^5(2z - 1)(z - 2)} \right) = -\frac{65}{24}.
\]

Then
\[
-\frac{1}{2i} \oint_C \frac{z^4 + 1}{z^5(2z - 1)(z - 2)} \, dz = -\frac{1}{2i} \left( \frac{21}{8} - \frac{65}{24} \right) = \frac{\pi}{12},
\]
as required.

14. Show that
\[
\int_0^{2\pi} \frac{d\theta}{(5 - 3 \sin \theta)^3} = \frac{5\pi}{32}.
\]

Letting \( z = e^{i\theta} \), we have \( \sin \theta = (z - z^{-1})/2i \), \( dz = id\theta \), \( dx = i dx \), and so
\[
\int_0^{2\pi} \frac{d\theta}{(5 - 3 \sin \theta)^3} = \oint_C \frac{dz}{(5 - 3(z - z^{-1})/2i)^3} = -\frac{4i}{3} \oint_C \frac{z \, dz}{(3z^2 - 10z - 8)^3}
\]
where \( C \) is the contour of Fig. 7-6.

The integrand has poles of order 2 at \( z = \frac{\sqrt{100} + 16}{6} = \frac{16i + 8i}{6} = 3i/2 \). Only the pole \( 3i/2 \) lies inside \( C \).

Residue at \( z = 3i/2 \) is
\[
\lim_{z \to 3i/2} \frac{d}{dz} \left( \frac{z^4 + 1}{(3z^2 - 10z - 8)^3} \right) = -\frac{5}{288}.
\]

Then
\[
-\frac{4i}{3} \oint_C \frac{z \, dz}{(3z^2 - 10z - 8)^3} = -\frac{4i}{3} \left( \frac{-5}{288} \right) = \frac{5\pi}{32}.
\]

Another method.

From Problem 12, we have for \( a > |b| \),
\[
\int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.
\]

Then by differentiating both sides with respect to \( a \) (considering \( b \) as constant) using Leibniz's rule, we have
\[
\frac{d}{da} \int_0^{2\pi} \frac{d\theta}{a + b \sin \theta} = \int_0^{2\pi} \frac{\partial}{\partial a} \left( \frac{1}{a + b \sin \theta} \right) \, d\theta = -\int_0^{2\pi} \frac{d\theta}{(a + b \sin \theta)^2}
\]
\[
= \frac{d}{da} \left( \frac{2\pi}{\sqrt{a^2 - b^2}} \right) = \frac{-2\pi a}{(a^2 - b^2)^{3/2}}
\]
i.e.,
\[
\int_0^{2\pi} \frac{d\theta}{(a + b \sin \theta)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}.
\]

Letting \( a = 5 \) and \( b = -3 \), we have
\[
\int_0^{2\pi} \frac{d\theta}{(5 - 3 \sin \theta)^2} = \frac{5\pi}{32}.
\]

DEFINITE INTEGRALS OF THE TYPE \( \int_0^{2\pi} F(x) \left\{ \begin{array}{c} \cos mx \\ \sin mx \end{array} \right\} \, dx \)

15. If \( |F(x)| \leq \frac{M}{|x|^k} \) for \( x = Re^{i\theta} \) where \( k > 0 \) and \( M \) are constants, prove that
\[
\lim_{n \to \infty} \int_0^{2\pi} e^{inx} F(x) \, dx = 0
\]
where \( R \) is the semicircular arc of Fig. 7-5 and \( n \) is a positive constant.

If \( x = Re^{i\theta} \), \( \int_0^{2\pi} e^{inx} F(x) \, dx = \int_0^{\pi} e^{inx} F(Re^{i\theta}) \, dRe^{i\theta} \). Then
\[ \int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta}) \, iRe^{i\theta} \, d\theta = \int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta}) \, |Re^{i\theta}| \, d\theta \]
\[ = \int_0^\pi e^{imR \cos \theta - mR \sin \theta} F(Re^{i\theta}) \, |Re^{i\theta}| \, d\theta \]
\[ = \int_0^\pi e^{-mR \sin \theta} |F(Re^{i\theta})| \, R \, d\theta \]
\[ \leq \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-mR \sin \theta} \, d\theta \]

Now \( \sin \theta \geq 2\theta/\pi \) for \( 0 \leq \theta \leq \pi/2 \), as can be seen geometrically from Fig. 7-7 or analytically from Prob. 99.

Then the last integral is less than or equal to
\[ \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-2mR\theta} \, d\theta = \frac{2M}{mR^k} (1 - e^{-mR}) \]

As \( R \to \infty \), this approaches zero, since \( m \) and \( k \) are positive, and the required result is proved.

16. Show that \[ \int_0^\infty \frac{\cos mx}{x^2 + 1} \, dx = \frac{\pi}{2} e^{-m}, \quad m > 0. \]

Consider \( \oint_C \frac{e^{imx}}{x^2 + 1} \, dx \) where \( C \) is the contour of Fig. 7-5. The integrand has simple poles at \( z = \pm i \), but only \( z = i \) lies inside \( C \).

Residue at \( z = i \) is \[ \lim_{x \to i} \frac{e^{imx}}{x^2 + 1} = \frac{e^{-m}}{2i}. \] Then
\[ \oint_C \frac{e^{imx}}{x^2 + 1} \, dx = 2\pi i \left( \frac{e^{-m}}{2i} \right) = \pi e^{-m} \]
or
\[ \int_{-R}^R \frac{e^{imx}}{x^2 + 1} \, dx + \int_0^\infty \frac{e^{imx}}{x^2 + 1} \, dx = \pi e^{-m} \]
i.e.,
\[ \int_{-R}^R \cos mx \, dx + i \int_{-R}^R \sin mx \, dx + \int_0^\infty \frac{e^{imx}}{x^2 + 1} \, dx = \pi e^{-m} \]
and so
\[ 2 \int_0^\infty \frac{\cos mx}{x^2 + 1} \, dx = \pi e^{-m} \]

Taking the limit as \( R \to \infty \) and using Problem 16 to show that the integral around \( C \) approaches zero, we obtain the required result.

17. Evaluate \( \int_{-\infty}^\infty \frac{x \sin \pi x}{x^2 + 2x + 5} \, dx \).

Consider \( \oint_C \frac{e^{ix\pi x}}{x^2 + 2x + 5} \, dx \) where \( C \) is the contour of Fig. 7-5. The integrand has simple poles at \( z = -1 \pm 2i \), but only \( z = -1 + 2i \) lies inside \( C \).

Residue at \( z = -1 + 2i \) is \[ \lim_{x \to -1 + 2i} \left( \frac{x + 1 - 2i}{x^2 + 2x + 5} \right) = \frac{e^{-2\pi}}{4i}. \] Then
\[ \oint_C \frac{e^{ix\pi x}}{x^2 + 2x + 5} \, dx = 2\pi i (-1 + 2i) \left( \frac{e^{-2\pi}}{4i} \right) = \frac{\pi}{2} (1 - 2i) e^{-2\pi} \]
or
\[ \int_{-R}^R \frac{e^{ix\pi x}}{x^2 + 2x + 5} \, dx + \int_{-R}^R \frac{e^{ix\pi x}}{x^2 + 2x + 5} \, dx = \frac{\pi}{2} (1 - 2i) e^{-2\pi} \]
i.e.,
\[ \int_{-R}^R \frac{\cos \pi x}{x^2 + 2x + 5} \, dx + i \int_{-R}^R \frac{\sin \pi x}{x^2 + 2x + 5} \, dx + \int_{-R}^R \frac{e^{ix\pi x}}{x^2 + 2x + 5} \, dx = \frac{\pi}{2} (1 - 2i) e^{-2\pi} \]
Residue of \( \tfrac{\pi \cot \pi z}{f(z)} \) at \( z = n, \ n = 0, \pm 1, \pm 2, \ldots \), is

\[
\lim_{z \to n} (z-n) \tfrac{\pi \cot \pi z}{f(z)} = \lim_{z \to n} \left( \tfrac{z-n}{\sin \pi z} \right) \cot \pi z f(z) = f(n)
\]

using L'Hopital's rule. We have assumed here that \( f(z) \) has no poles at \( z = n \), since otherwise the given series diverges.

By the residue theorem,

\[
\oint_{C_N} \tfrac{\pi \cot \pi z}{f(z)} \, dz = \sum_{n=-N}^{N} f(n) + S \tag{1}
\]

where \( S \) is the sum of the residues of \( \tfrac{\pi \cot \pi z}{f(z)} \) at the poles of \( f(z) \). By Problem 24 and our assumption on \( f(z) \), we have

\[
\left| \oint_{C_N} \tfrac{\pi \cot \pi z}{f(z)} \, dz \right| \leq \frac{\pi A M}{N^3} (8N + 4)
\]

since the length of path \( C_N \) is \( 8N + 4 \). Then taking the limit as \( N \to \infty \) we see that

\[
\lim_{N \to \infty} \oint_{C_N} \tfrac{\pi \cot \pi z}{f(z)} \, dz = 0 \tag{2}
\]

Thus from (1) we have as required,

\[
\sum_{n=-\infty}^{\infty} f(n) = -S \tag{3}
\]

Case 2: \( f(z) \) has infinitely many poles.

If \( f(z) \) has an infinite number of poles, we can obtain the required result by an appropriate limiting procedure. See Problem 105.

26. Prove that \( \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a \) where \( a > 0 \).

Let \( f(z) = \frac{1}{z^2 + a^2} \) which has simple poles at \( z = \pm ai \).

Residue of \( \tfrac{\pi \cot \pi z}{z^2 + a^2} \) at \( z = ai \) is

\[
\lim_{z \to ai} \frac{\pi \cot \pi z}{(z-ai)(z+ai)} = \frac{\pi \cot \pi ai}{2ai} = -\frac{\pi}{2a} \coth \pi a
\]

Similarly the residue at \( z = -ai \) is \( \frac{\pi}{2a} \coth \pi a \), and the sum of the residues is \( -\frac{\pi}{a} \coth \pi a \). Then by Problem 25,

\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -\text{(sum of residues)} = \frac{\pi}{a} \coth \pi a
\]

27. Prove that \( \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth \pi a - \frac{1}{2a^2} \) where \( a > 0 \).

The result of Problem 26 can be written in the form

\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a
\]

or

\[
2 \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{a^2} = \frac{\pi}{a} \coth \pi a
\]

which gives the required result.
31. Prove that if \( \alpha \neq 0, \pm 1, \pm 2, \ldots \), then

\[
\frac{\alpha^2 + 1}{(\alpha^2 - 1)^2} - \frac{\alpha^2 + 4}{(\alpha^2 - 4)^2} + \frac{\alpha^2 + 9}{(\alpha^2 - 9)^2} - \cdots = \frac{1}{2\alpha^2} - \frac{\pi^2 \cos \pi \alpha}{2 \sin^2 \pi \alpha}
\]

The result of Problem 30 can be written in the form

\[
\frac{1}{\alpha^2} - \frac{1}{(\alpha + 1)^2} + \frac{1}{(\alpha - 1)^2} + \frac{1}{(\alpha + 2)^2} - \frac{1}{(\alpha - 2)^2} + \cdots = \frac{\pi^2 \cos \pi \alpha}{\sin^2 \pi \alpha}
\]

or

\[
\frac{1}{\alpha^2} - \frac{2(\alpha^2 + 1)}{(\alpha^2 - 1)^2} + \frac{2(\alpha^2 + 4)}{(\alpha^2 - 4)^2} - \frac{2(\alpha^2 + 9)}{(\alpha^2 - 9)^2} + \cdots = \frac{\pi^2 \cos \pi \alpha}{\sin^2 \pi \alpha}
\]

from which the required result follows. Note that the grouping of terms in the infinite series is permissible since the series is absolutely convergent.

32. Prove that

\[
\frac{1}{z^2} - \frac{1}{2z} + \frac{1}{5z} - \frac{1}{7z} + \cdots = \frac{\pi^2}{32}.
\]

We have

\[
F(z) = \frac{\pi \sec \pi z}{z^2} = \frac{\pi}{z^2 \cos \pi z} = \frac{\pi}{z^2 (1 - z^2/2! + \cdots)}
\]

so that the residue at \( z = 0 \) is \( \pi^2/2 \).

The residue of \( F(z) \) at \( z = n \pm \frac{1}{2}, n = 0, \pm 1, \pm 2, \ldots \) (which are the simple poles of \( \sec \pi z \)), is

\[
\lim_{z \to n \pm \frac{1}{2}} (z - (n \pm \frac{1}{2})) \frac{\pi}{(z - n \pm \frac{1}{2})^2 \cos \pi z} = \frac{\pi}{n \pm \frac{1}{2}} \lim_{z \to n \pm \frac{1}{2}} \frac{z - (n \pm \frac{1}{2})}{\cos \pi z} = \frac{(-1)^n}{(n \pm \frac{1}{2})^2}
\]

If \( C_N \) is a square with vertices at \( N(1 + 0), N(1 - 0), N(-1 + i), N(-1 - i) \), then

\[
\oint_{C_N} \frac{\pi \sec \pi z}{z^2} \, dz = -\pi \sum_{n=-N}^{N} \frac{(-1)^n}{(n \pm \frac{1}{2})^2} + \frac{\pi^2}{2} = -8 \sum_{n=-N}^{N} \frac{(-1)^n}{(2n+1)^2} + \frac{\pi^2}{2}
\]

and since the integral on the left approaches zero as \( N \to \infty \), we have

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 2 \left( \frac{1}{13} - \frac{1}{35} + \frac{1}{53} - \cdots \right) \approx \frac{\pi^2}{16}
\]

from which the required result follows.

**MITTAG-LEFFLER’S EXPANSION THEOREM**

33. Prove Mittag-Leffler’s expansion theorem (see Page 176).

Let \( f(z) \) have poles at \( z = a_n, n = 1, 2, \ldots \), and suppose that \( z = \zeta \) is not a pole of \( f(z) \). Then the function \( f(z) \) at \( z = a_n, n = 1, 2, \ldots \) and \( \zeta \).

Residue of \( f(z) \) at \( z = a_n, n = 1, 2, 3, \ldots \), is

\[
\lim_{z \to a_n} (z - a_n) f(z) = \frac{b_n}{a_n - \zeta}.
\]

Residue of \( f(z) \) at \( z = \zeta \) is

\[
\lim_{z \to \zeta} (z - \zeta) \frac{f(z)}{z - \zeta} = f(\zeta).
\]

Then by the residue theorem,

\[
\frac{1}{2\pi i} \oint_{C} f(z) \, dz = f(\zeta) + \sum_{n} \frac{b_n}{a_n - \zeta}
\]

where the last summation is taken over all poles inside circle \( C \) of radius \( R_N \) (Fig. 7-14).

Suppose that \( f(z) \) is analytic at \( z = 0 \). Then putting \( \zeta = 0 \) in (1), we have

\[
\frac{1}{2\pi i} \oint_{C} f(z) \, dz = f(0) + \sum_{n} \frac{b_n}{a_n}
\]

Fig. 7-14