Recursive Equations

Analysis of recursive algorithms results in recursive equations. If $T(n)$ is the time taken to find the max element in an array of size $n$, then:

\begin{itemize}
  \item $T(n) = c_1$, if $n = 1$
  \item $T(n) = T(n - 1) + c_2$, otherwise
\end{itemize}

The closed-form solution is $\cdots$
Recursive Equations

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The closed-form solution is ⋅⋅⋅

$T(n) = c_2(n-1) + c_1$
Recursive Equations

- At times recursive equations can be tricky.
- **Master Theorem** is used to solve recurrence equations in asymptotic terms.
Recurrence relations can also be used to define a sequence

- Example: $x_{n+1} = c \cdot x_n$, given ($n \geq 0; x_0 = 1$)

The closed-form solution is ...
Recurrence relations can also be used to define a sequence

Example: $x_{n+1} = c \cdot x_n$, given ($n \geq 0; x_0 = 1$)

The closed-form solution is ...

$x_n = c^n$
Recursive Equations

A slightly different one: \( x_{n+1} = b_{n+1} \cdot x_n \)

Given:

- \( n \geq 0 \)
- the value of \( x_0 \)
- the set \( \{ b_1, b_2, \ldots \} \)

The closed-form solution is ...
Recursive Equations

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Given:
- \( n \geq 0 \)
- the value of \( x_0 \)
- the set \( \{b_1, b_2, \ldots \} \)

The closed-form solution is ...
\[
x_n = b_n \cdot b_{n-1} \cdot b_{n-2} \cdots b_1 \cdot x_0
\]
Recursive Equations

Let us raise the ante further: \( x_{n+1} = b_{n+1}x_n + c_{n+1} \)

Given:

- \( n \geq 0 \)
- the value of \( x_0 \)
- the set \( \{b_1, b_2, \cdots \} \)
- the set \( \{c_1, c_2, \cdots \} \)
Recursive Equations

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**Given:**
- \(n \geq 0\)
- the value of \(x_0\)
- the set \(\{b_1, b_2, \cdots\}\)
- the set \(\{c_1, c_2, \cdots\}\)

**Hint:** Reduce it to the form \(y_{n+1} = y_n + d_{n+1}\)
Example: $x_{n+1} = 3x_n + n$, where $(n \geq 0; x_0 = 0)$

If $x_n = 3^n y_n$ then we get

$$y_{n+1} = y_n + n/3^{n+1}, \text{ where } n \geq 0; y_0 = 0$$
Example: \( x_{n+1} = 3 \cdot x_n + n \), where \( (n \geq 0; x_0 = 0) \)

If \( x_n = 3^n \cdot y_n \) then we get

\[
y_{n+1} = y_n + \frac{n}{3^{n+1}}, \text{ where } n \geq 0; y_0 = 0
\]

Finally we get: \( x_n = 3^n \sum_{j=1}^{n-1} \frac{j}{3^{j+1}} \)
Recursive Equations

Till now we have only considered first-order recursive equations.

Note:

- In first-order recursive equations, the current value depends only on the previous value.
- In second-order recursive equations, the current value depends on the previous two values.
Recursive Equations

Let the equation be: $x_{n+1} = a.x_n + b.x_{n-1}$

Given:

- $n \geq 1$
- the value of $x_0, x_1$ and $a, b$
Let the equation be: $x_{n+1} = a \cdot x_n + b \cdot x_{n-1}$

Given:

- $n \geq 1$
- the value of $x_0, x_1$ and $a, b$

Hint: Call for a trial solution (as you solve second-order differential equations)
Recursive Equations

Let the equation be: \( x_{n+1} = a \cdot x_n + b \cdot x_{n-1} \)

Follow the following steps:

- Let the trial solution be \( x_n = \alpha^n \), and substitute it in the given equation
- We obtain the quadratic equation \( \alpha^2 = a \cdot \alpha + b \)
- If \( \alpha_+ \) and \( \alpha_- \) are the distinct roots, the general solution is \( x_n = c_1 \cdot \alpha_+^n + c_2 \cdot \alpha_-^n \)
- The constants \( c_1 \) and \( c_2 \) will be determined so that \( x_0, x_1 \) have the assigned values.
Example

$L_0 = 100000, L_1 = 200000$, and $L_n = (L_{n-1} + L_{n-2})/2$

- The characteristic polynomial is \( \cdots \)
- The roots are \( \cdots \)
- The general solution is \( \cdots \)
- The values of $c_1$ and $c_2$ are \( \cdots \)
Example

\[ L_0 = 100000, \quad L_1 = 200000, \quad \text{and} \quad L_n = \frac{(L_{n-1} + L_{n-2})}{2} \]

- The characteristic polynomial is \( x^2 - x/2 - 1/2 \)
- The roots are 1 and \(-1/2\)
- The general solution is \( L_n = c_1 + c_2 (-1/2)^n \)
- \( c_1 = \frac{500000}{3} \) and \( c_2 = -\frac{200000}{3} \).