Asymptotic notations and the growth of functions

Some preliminary notions

1. A set is a collection of distinct objects.

2. The Cartesian product of two sets $A, B$ is the set of all ordered pairs, where the first element of the pairs is drawn from the first set, and the second from the second set. $A \times B = \{(a, b) | a \in A, b \in B\}$. Thus, the cartesian product operator is not commutative in general.

3. A relation between a pair of sets (in a given order) is any subset of their cartesian product. A relation on a set, is any subset of its cartesian product with itself.

4. A reflexive relation on a set $A$, is any relation which contains all ordered pairs of the form $(a, a), \forall a \in A$.

5. A symmetric relation on a set $A$, is any relation which contains the ordered pair $(a, b)$ if and only if it contains the ordered pair $b, a$, for every pair of elements $a, b \in A$.

6. A transitive relation on a set $A$, is any relation where the presence of $(a, b)$ and $(b, c)$ implies the presence of $(a, c)$.

7. An equivalence relation on a set $A$, is any relation which is reflexive, symmetric and transitive. An equivalence relation on a set induces a corresponding partition of the underlying set.

8. A partial order is a relation on a set which is reflexive, antisymmetric and transitive. A total order is a partial order where each pair of elements in the underlying set are present as an ordered pair once.

9. A function from a set $A$ to a set $B$ is any relation, such that every element in $A$ occurs in exactly one ordered pair (as the first element
of course). Thus each element in $A$, has a unique image in $B$. The set $A$ is called the domain of definition of the function, and the set $B$ is called the codomain. The actual elements of $B$, which are images of at least one element of $A$ in the function constitute the range of the function. It is customary to write $f : A \rightarrow B$.

Throughout the rest of this document, it is assumed that all functions referred to are asymptotically non-negative (that is, they take non-negative values for all values of their input argument, beyond greater than a threshold).

Definitions of the asymptotic notations

1. A function $f_1(n)$ is said to be $o(f_2(n))$, where $f_2$ is another function, if for all $c > 0$, there exists a corresponding $n_0$, such that $f_1(n) < cf_2(n)$, $\forall n \geq n_0$.

2. A function $f_1(n)$ is said to be $O(f_2(n))$, where $f_2$ is another function, if there exist constants $c > 0$, and $n_0 \in \mathbb{N}$ such that, $f_1(n) \leq cf_2(n)$, $\forall n \geq n_0$.

3. A function $f_1(n)$ is said to be $\theta(f_2(n))$, where $f_2$ is another function, if there exist constants $c_1, c_2 > 0$, and $n_0 \in \mathbb{N}$, such that $c_1f_1(n) \leq f_2(n) \leq c_2f_1(n)$, $\forall n \geq n_0$.

4. A function $f_1(n)$ is said to be $\Omega(f_2(n))$, where $f_2$ is another function, if there exist constants $c > 0$, and $n_0 \in \mathbb{N}$ such that, $f_1(n) \geq cf_2(n)$, $\forall n \geq n_0$.

5. A function $f_1(n)$ is said to be $\omega(f_2(n))$, where $f_2$ is another function, if for all $c > 0$, there exists a corresponding $n_0$, such that $f_1(n) > cf_2(n)$, $\forall n \geq n_0$.

Some properties of these asymptotic properties

1. The five asymptotic notations for relating pairs of functions, i.e. $o, O, \theta, \Omega, \omega$ are analogous to $<, \leq, =, \geq, >$ for comparing pairs of numbers.

2. Unlike those operators over real numbers, it may occasionally be impossible to compare a pair of functions under these asymptotic notations. An example of a pair of functions which cannot be related by the above is: $n^2$ and $n^{2+\sin n}$.
3. The set of functions in the class $\theta(f(n))$ is precisely the intersection of the functions in the classes $O(f(n))$ and $\Omega(f(n))$. Thus, $\theta(f(n)) = O(f(n)) \cap \Omega(f(n))$.

4. The classes of functions $o(f(n))$ and $\omega(f(n))$ are disjoint. Equivalently, $o(f(n)) \cap \omega(f(n)) = \emptyset$.

**Alternative (equivalent) definition of the asymptotic notations**

1. $f_1(n)$ is $o(f_2(n))$, if $\lim_{n \to \infty} \frac{f_1(n)}{f_2(n)} = 0$.

2. $f_1(n)$ is $O(f_2(n))$, if $\lim_{n \to \infty} \frac{f_1(n)}{f_2(n)}$ exists and is not equal to $\infty$.

3. $f_1(n)$ is $\theta(f_2(n))$, if $\lim_{n \to \infty} \frac{f_1(n)}{f_2(n)} = c \neq 0$.

4. $f_1(n)$ is $\Omega(f_2(n))$, if $\lim_{n \to \infty} \frac{f_1(n)}{f_2(n)} = c \neq 0$ or $\lim_{n \to \infty} \frac{f_1(n)}{f_2(n)} = \infty$.

5. $f_1(n)$ is $\omega(f_2(n))$, if $\lim_{n \to \infty} \frac{f_1(n)}{f_2(n)} = \infty$.

**Standard functions**

1. Some standard functions are constant functions, logarithmic functions, poly-logarithmic functions, polynomial functions, exponential functions and super-exponential functions.

2. The above mentioned functions were stated in their relative orders of growth.

3. (Poly)logarithmic functions are of the form $(\log n)^c$, where $c$ is a constant independent of $n$.

4. Polynomial functions are of the order of growth of $n^c$, where $c$ is a constant independent of $n$.

5. Exponential functions are functions of the order of growth of $a^c$, where $a > 0$ is a constant independent of $n$.

6. Super-exponential functions are those in the class $\omega(a^n)$, for any constant $a$.

7. In general, while trying to compare a set of functions pairwise, to determine relative growth rate, it is a good idea to use these broad classes to refine the set, before proceeding to compare functions within the same class.
8. The *iterated logarithmic function*, denoted $\log^* n$, is the number of times the logarithmic function needs to be applied in a pipelined fashion starting with the argument $n$, till the value becomes zero or less.

A big example with several functions to sort out.

<table>
<thead>
<tr>
<th>$\log(\log^* n)$</th>
<th>$2^{\log^* n}$</th>
<th>$(\sqrt{2})^{\log n}$</th>
<th>$n^2$</th>
<th>$n!$</th>
<th>$(\log n)!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\frac{3}{2})^n$</td>
<td>$n^3$</td>
<td>$\log^2 n$</td>
<td>$\log (n!)$</td>
<td>$2^{2n}$</td>
<td>$\frac{1}{n^{\log n}}$</td>
</tr>
<tr>
<td>$\ln \ln n$</td>
<td>$\log^* n$</td>
<td>$n^{2n}$</td>
<td>$n^{\log \log n}$</td>
<td>$\ln n$</td>
<td>$1$</td>
</tr>
<tr>
<td>$2^{\log n}$</td>
<td>$(\log n)^{\log n}$</td>
<td>$e^n$</td>
<td>$4^{\log n}$</td>
<td>$(n + 1)!$</td>
<td>$\sqrt{\log n}$</td>
</tr>
<tr>
<td>$\log^* \log n$</td>
<td>$2^{\sqrt{2}^{\log n}}$</td>
<td>$n$</td>
<td>$2^n$</td>
<td>$n^{\log n}$</td>
<td>$2^{2^{n+1}}$</td>
</tr>
</tbody>
</table>

Solution

1. $1, n^{\frac{1}{\log n}}$
2. $\log(\log^* n)$
3. $\log^* n, \log^* \log n$
4. $2^{\log^* n}$
5. $\ln \ln n$
6. $\sqrt{\log n}$
7. $\ln n$
8. $\log^2 n$
9. $2^{\sqrt{2}^{\log n}}$
10. $(\sqrt{2})^{\log n}$
11. $n, 2^{\log n}$
12. $n^{\log n}, \log(n!)$
13. $n^2, 4^{\log n}$
14. $n^3$
15. $(\log n)!$
16. $n^{\log \log n}, (\log n)^{\log n}$
17. $(\frac{3}{2})^n$
18. $2^n$
19. $n2^n$
20. $e^n$
21. $n!$
22. $(n + 1)!$
23. $2^{2^n}$
24. $2^{2^{n+1}}$

In the above solution functions in entry 1 are constant, those in 2-4 are much smaller than logarithmic functions, 5 is double logarithmic, 6-8 are poly-logarithmic, 9 is between poly-logarithmic and polynomial, 10-14 are polynomial, 15-16 are super-polynomial but subexponential, 17-20 are exponential, 21-22 are super-exponential but not double exponential, 23-24 are double exponential. Standard methods for resolving pairs of such functions, are substitutions, taking logarithms or functional compositions.