Capacity of MIMO Systems in Rayleigh Fading and Shadowing

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Abstract— **In this paper, we compare the capacity of multiple-input and multiple-output (MIMO) systems under the dual effects of Rayleigh fading and log-normal shadowing with that of Rayleigh fading alone. Our numerical results suggest that in general, there could be a loss in capacity with both optimal and sub-optimal adaptive transmission schemes under the combined effect of fading and shadowing when compared to that of a fading alone case.**

I. Introduction

The multiple-input and multiple-output (MIMO) systems can achieve large spectral efficiencies in wireless links by exploiting channel fading as shown in the pioneering works of Foschini and Telatar in [1] and [2]. For example, in [2] it was shown that when channel state information (CSI) is available at both transmitter and receiver the capacity could be achieved by using the spatial water-filling (SWF) algorithm. Also, for a time varying single antenna system (i.e. where both transmitter and receiver are equipped with single antenna) with CSI available at both transmitter and receiver and under an average power constraint the optimal adaptation policy was shown to be water-pouring in time [3]. The optimal adaptation policy could be complex to implement and hence two sub-optimal adaptive transmission schemes named channel inversion (CI) and truncated channel inversion (TCI) were also proposed in [3]. Recently the optimal adaptation policy proposed in [3] was extended to a MIMO system in Rayleigh fading by Jayaweera and Poor in [4]. The optimal adaptation policy in this case was shown to be the space-time water-filling (STWF) algorithm. The sub-optimal schemes proposed in [3] were extended to a MIMO system in [5] and [6], again assuming Rayleigh fading.

In this paper, we consider comparison of capacity of MIMO systems under the combined effect of Rayleigh fading and log-normal shadowing with that of fading alone. We provide expressions for evaluating the capacity with SWF, STWF, CI and TCI schemes. Our numerical results suggest that, in general the presence of fading and shadowing together could lead to a reduction in capacity.

The rest of this paper is organized as follows: Section II describes the system model. Section III considers the capacity of MIMO systems under the combined effect of fading and shadowing. Section IV presents numerical results and Section V contains conclusions.

Notation

The superscript ^{H} denotes complex conjugate transpose. The notations $\mathrm{tr}(\mathbf{A})$ and $\det(\mathbf{A})$ denote the trace and determinant of the matrix $\mathbf{A}$ respectively. Expectation with respect to the distribution of the random variable γ is denoted by $\mathbb{E}_\gamma\{.\}$. The notation $\mathbf{A}_{i,i}$ denotes the diagonal element of matrix $\mathbf{A}$. The notation $\mathbf{I}_n$ denotes an identity matrix of size $n \times n$. The

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notation a^+ denotes $\max(0,a)$. Also throughout this paper we denote vectors in bold faced small letters and matrices in bold faced capitals.

II. System Model Description

We consider a flat-fading communication link in which transmitter and receiver are equipped with N_T and N_R antennas respectively. In general the discrete-time received signal vector in such a system can be written as

$$\mathbf{y}(i) = \mathbf{H}(i)\mathbf{x}(i) + \mathbf{n}(i) \ , \tag{1}$$

where $\mathbf{y}$ is the received signal vector of dimension $N_R \times 1$, i denotes the time instant, $\mathbf{H}$ denotes the channel matrix of size $N_R \times N_T$, $\mathbf{x}$ denotes the transmitted signal vector of dimension $N_T \times 1$ and $\mathbf{n}$ denotes the receiver noise vector of dimension $N_R \times 1$ which is assumed to be spatially and temporally white with a variance of N_0.

For the present case of Rayleigh fading plus shadowing we model the channel matrix $\mathbf{H}$ as follows:

$$\mathbf{H} = \sqrt{\frac{s}{\zeta}}\mathbf{H}_w \ , \tag{2}$$

where s is a log-normal random variable which models the shadowing effect, $\zeta = \mathbb{E}_s[s]$ and $\mathbf{H}_w$ is a matrix of size $N_R \times N_T$ which contains zero mean circularly symmetric complex Gaussian random variables of variance $\frac{1}{2}$ per dimension. Furthermore, we assume the elements of $\mathbf{H}_w$ to be ergodic and stationary and are independent of s. The model (2) assumes that shadowing occurs on large spatial scales and hence it affects all receiver antenna elements in a similar manner [7]. Note that normalization in (2) ensures proper comparison between a Rayleigh fading plus shadowing channel and a Rayleigh fading alone channel. For the analysis throughout this paper, we assume CSI is available at both transmitter and receiver and the transmitter is subjected to an average power constraint i.e. $\mathbb{E}\left\{\mathbf{x}^H\mathbf{x}\right\} = P$.

Note that, since s is log-normal (i.e. $10\log_{10} s \sim \mathcal{N}\left(0,\rho^2\right)$) the probability density function (pdf) of s can be written as follows:

$$r(s) = \frac{10}{\ln 10\sqrt{2\pi}\rho}\frac{1}{s}e^{-\frac{(10\log_{10} s)^2}{2\rho^2}} \ . \tag{3}$$

For our future use we also define $n = \max\{N_R, N_T\}$ and $m = \min\{N_R, N_T\}$.

III. Capacity of MIMO Systems under Fading and Shadowing

A. Capacity with Spatial Water-Filling

This case was previously considered for the case of Rayleigh fading alone in [2]. Here power adaptation is performed with a total power constraint for each channel realization. The capacity maximization problem for a given $\mathbf{H}$ for this case can be formulated as [2]:

$$C_{SWF} = \max_{\mathbf{Q},\ \mathrm{tr}(\mathbf{Q})\leq P} \log_2 \det\left(\mathbf{I}_{N_R} + \frac{\mathbf{H}\mathbf{Q}\mathbf{H}^H}{N_0}\right) \ , \tag{4}$$

where $\mathbf{Q} = \mathbb{E}\left[\mathbf{x}\mathbf{x}^H\right]$, P is the total transmit power.

Applying the eigenvalue decomposition to $\mathbf{H}^H\mathbf{H}$, we can write $\mathbf{H}^H\mathbf{H} = \mathbf{U}^H\boldsymbol{\Upsilon}\mathbf{U}$, where $\mathbf{U}$ is a unitary matrix,

$$\boldsymbol{\Upsilon} = \mathrm{diag}\{t_1, \cdots, t_m\} \ . \tag{5}$$

Note that $t_i = s\lambda_i$, where λ_i's correspond to the eigenvalues of either $\mathbf{H}_w^H\mathbf{H}_w$ or $\mathbf{H}_w\mathbf{H}_w^H$ for $i = 1, \cdots, m$.

As pointed out in [2] the optimization in (4) can be carried out over $\widetilde{\mathbf{Q}} = \mathbf{U}\mathbf{Q}\mathbf{U}^H$ and the capacity achieving $\widetilde{\mathbf{Q}}$ can be shown to be diagonal. Thus (4) can be written as

$$C_{SWF} = \sum_{i=1}^{m}\log_2\left(1 + \frac{1}{N_0}\widetilde{\mathbf{Q}}_{i,i}t_i\right) \ , \tag{6}$$

where the optimal $\widetilde{\mathbf{Q}}_{i,i} = \left(\mu - \frac{N_0}{t_i}\right)^+$ and μ is solved iteratively to satisfy tr $\left(\widetilde{\mathbf{Q}}\right) = P$. Note that, here we assume $t_1 \geq t_2 \cdots \geq t_m$ and to obtain average capacity we average (4) over $\mathbf{H}$.

B. Capacity with Space-Time Water-Filling

Applying eigenvalue decomposition to the channel matrix $\mathbf{H}$ in (1) we can write (ignoring the time index i)

$$\widetilde{\mathbf{y}} = \boldsymbol{\Upsilon}\widetilde{\mathbf{x}} + \widetilde{\mathbf{n}} \ , \tag{7}$$

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where $\widetilde{\mathbf{y}}$ is the unitary transformation of the vector $\mathbf{y}$, $\boldsymbol{\Upsilon}$ is defined as in (5), $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{n}}$ are unitary transformations of the vectors $\mathbf{x}$ and $\mathbf{n}$ respectively.

Since we are assuming the transmitter is subjected to an average power constraint i.e. $\mathbb{E}\left\{\mathbf{x}^H\mathbf{x}\right\} = P$ which can be re-written using the transformed model in (7) as

$$\text{tr}\left[\mathbb{E}_{\boldsymbol{\Upsilon}}\left(\widetilde{\mathbf{Q}}\right)\right] = P \tag{8}$$

where we re-define $\widetilde{\mathbf{Q}} = \left\{\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^H\right\}$. Further, we assume that $\widetilde{\mathbf{Q}}$ to be a function of $\boldsymbol{\Upsilon}$ i.e. we adapt the transmitter power based on the channel matrix $\mathbf{H}$.

For our future use let us define the following $m \times m$ matrix,

$$\mathbf{W} = \begin{cases} \mathbf{H}_w\mathbf{H}_w^H \text{ if } N_R \leq N_T \\ \mathbf{H}_w^H\mathbf{H}_w \text{ if } N_R > N_T \end{cases} . \tag{9}$$

The distribution of $\mathbf{W}$ in (9) is given by the well-known complex central Wishart distribution.

Using the definition for the capacity of a vector, time-varying channel with the adaptive transmission scheme in [4], we can write the capacity with adaptive transmission scheme to be

$$C = \sum_{i=1}^{m}\mathbb{E}\left[\log_2\left(1+\frac{\widetilde{\mathbf{Q}}_{ii}}{(P/m)}\gamma_i\right)\right] , \tag{10}$$

where we define $\gamma_i = \overline{\gamma}t_i$ and $\overline{\gamma} = \frac{P}{mN_0\zeta}$. Note that, here λ_i's correspond to the eigenvalues of central Wishart matrix defined in (9).

Thus, for the present case of Rayleigh fading plus shadowing we need to consider the effective channel gain $t_i = s\lambda_i$ in order to evaluate the capacity in (10). In the following steps, we derive the pdf of unordered channel gain t and hence unordered γ.

Using the independence of s and λ, the pdf of unordered channel gain t can be written as

$$g(t) = \frac{10}{\ln 10\sqrt{2\pi}\rho}\int_0^\infty p\left(\frac{t}{s}\right)\frac{1}{s^2}e^{-\frac{(10\log_{10}s)^2}{2\rho^2}}ds , \tag{11}$$

where $p(\lambda)$ is the pdf of any unordered eigenvalue of central Wishart matrix obtained in [2] given by

$$p_\lambda(\lambda) = \frac{e^{-\lambda}\lambda^{n-m}}{m}\sum_{k=1}^{m}\frac{(k-1)!}{(n-m+k-1)!}\left[L_{k-1}^{n-m}(\lambda)\right]^2 , \tag{12}$$

where L_k^a is the associated Laguerre polynomial of order k defined as (for $k \geq 0$),

$$L_k^a(\lambda) = \frac{1}{k!}e^\lambda\lambda^{-a}\frac{d^k}{d\lambda^k}\left[e^\lambda\lambda^{a+k}\right] . \tag{13}$$

If $f_\gamma(\gamma)$ denotes the pdf of any unordered γ_i, for $i = 1, 2, \cdots, m$, then we have,

$$f(\gamma) = \frac{g\left(\frac{\gamma}{\overline{\gamma}}\right)}{\overline{\gamma}} . \tag{14}$$

With this the capacity with STWF can be shown as in [4] to be

$$C_{STWF} = m\int_{\gamma_0}^\infty \log_2\left(\frac{\gamma}{\gamma_0}\right)f(\gamma)\,d\gamma , \tag{15}$$

where the cutoff value γ_0 in (15) is obtained by solving the integral equation

$$\int_{\gamma_0}^\infty\left(\frac{1}{\gamma_0}-\frac{1}{\gamma}\right)f(\gamma)\,d\gamma = 1 . \tag{16}$$

C. Capacity with Channel Inversion

In the case of CI, the transmitter uses instantaneous CSI to keep the received signal-to-noise Ratio (SNR) per eigen-mode constant: i.e.,

$$\frac{\widetilde{\mathbf{Q}}_{i,i}\gamma_i}{P/m} = \sigma . \tag{17}$$

Using (8) and (17) we can show that,

$$\begin{aligned}\sigma &= \frac{1}{\mathbb{E}_\gamma\left[\frac{1}{\gamma}\right]} , \\ &= \frac{\overline{\gamma}}{\mathbb{E}_s\left[\frac{1}{s}\right]\mathbb{E}_\lambda\left[\frac{1}{\lambda}\right]} .\end{aligned} \tag{18}$$

Using equation 7.414.12 of [8], applying a transformation formula for hypergeometric function (equation 9.134.3 of [8]) and using the definition of hypergeometric series (equation 9.100) in [8] it is straightforward to show that

$$\mathbb{E}_\lambda\left[\frac{1}{\lambda}\right] = \frac{1}{n-m}.$$

Thus (18) can be written as

$$\sigma = \overline{\gamma}(n-m)\sigma_1 , \tag{19}$$

where σ_1 is defined as

$$\sigma_1 = \frac{1}{\mathbb{E}_s\left[\frac{1}{s}\right]} . \tag{20}$$

Using (19) we can show the capacity with CI to be

$$C_{CI} = m\log_2\left(1+\overline{\gamma}(n-m)\sigma_1\right) . \tag{21}$$

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D. Capacity with Truncated Channel Inversion

The method of channel inversion suffers from severe capacity penalty especially when the number of diversity branches are less. Also note from (21) that for the case of equal number of antennas at both transmitter and receiver i.e. for $n = m$, the capacity with CI becomes zero. For such cases, TCI offers higher capacity.

The adaptation rule for TCI can be stated as follows:

$$\frac{\widetilde{\mathbf{Q}}_{i,i}}{\frac{P}{m}} = \begin{cases} \frac{\sigma_{TCI}}{\gamma} \text{ if } \gamma \geq \gamma_0 \\ 0 \text{ if } \gamma < \gamma_0 \end{cases} , \quad (22)$$

where σ_{TCI} for the case of TCI can be shown to be

$$\sigma_{TCI} = \frac{1}{\mathbb{E}_\gamma\left[\frac{1}{\gamma}\right]} . \quad (23)$$

Thus, for the case of TCI capacity can be written as

$$C_{TCI} = m \log_2\left(1 + \sigma_{TCI}\right) \; P(\gamma \geq \gamma_0) . \quad (24)$$

IV. Numerical Results

In all numerical results we have set ρ equal to 8 dB and we assume $N_T = 4$ and $N_R = 6$. Figure 1 shows the capacity comparison of the MIMO system with SWF and STWF under the dual effect of fading and shadowing with that of a fading alone case. The results show that in general the presence of combined fading and shadowing could lead to a reduction in capacity. An interesting result that can be observed from Fig. 1 is that in the case of fading plus shadowing the capacity gap between the SWF and STWF is larger compared to that of fading alone case.

Figure 2 shows the comparison of capacity of MIMO systems with CI and TCI under the dual effects of fading and shadowing with that of fading alone case. The results show that the capacity with CI and TCI under the combined effect of fading and shadowing is much less than that of fading alone case. The CI scheme suffers from the worst penalty due to shadowing. In all the cases involving TCI we have set $\frac{\gamma_0}{\bar{\gamma}} = 0.5$.

Figure 3 shows the plot of normalized channel gain for the fading alone and fading plus shadowing cases. Figures

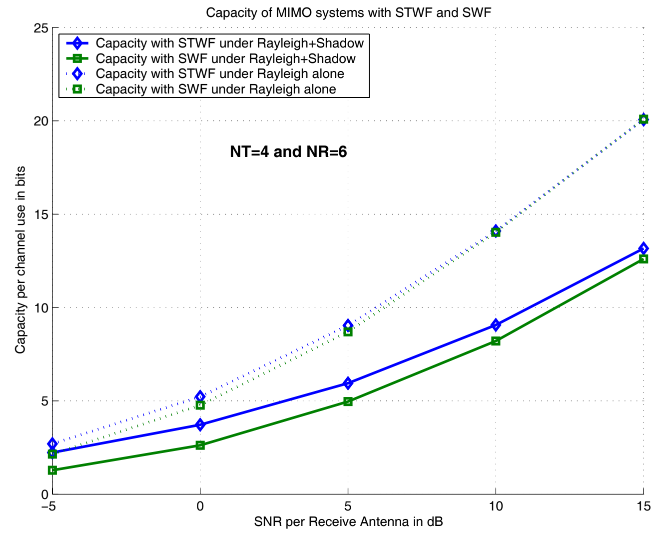

Fig. 1. Capacity of MIMO systems with STWF and SWF.

(1) and (2) show that in general the presence of fading plus shadowing could lead to a reduction in capacity for the optimal (SWF, STWF) and sub-optimal (CI, TCI) schemes. This can be attributed to the fact that, in the initial portion of the curve (i.e. between 0 and 8) pure fading channel has got a higher probability compared to the fading plus shadowing case. But in the later portion of the curve (i.e. beyond 10) fading plus shadowing channel has got higher probability compared to the fading alone case, but these probability values are too small to give any significant capacity gain with the fading plus shadowing case.

V. Conclusions

In this paper we investigated the capacity of MIMO systems under the dual effects of fading and shadowing. Our numerical results suggest that in general the presence of fading and shadowing could lead to a reduction in capacity with all adaptive schemes considered.

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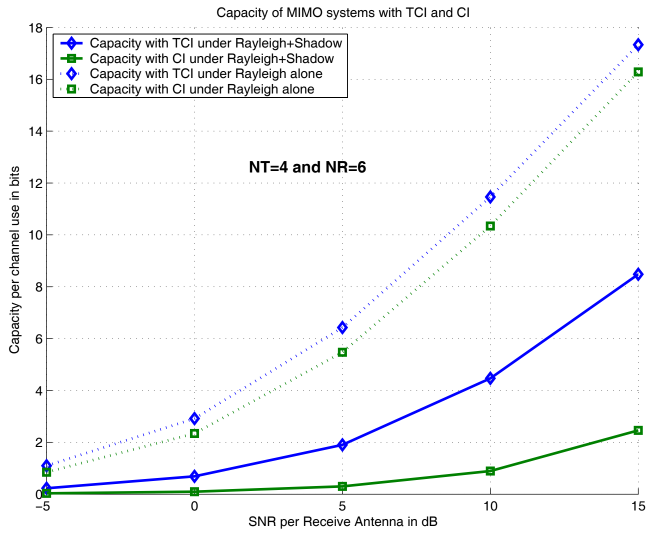

Fig. 2. Capacity of MIMO systems with CI and TCI.

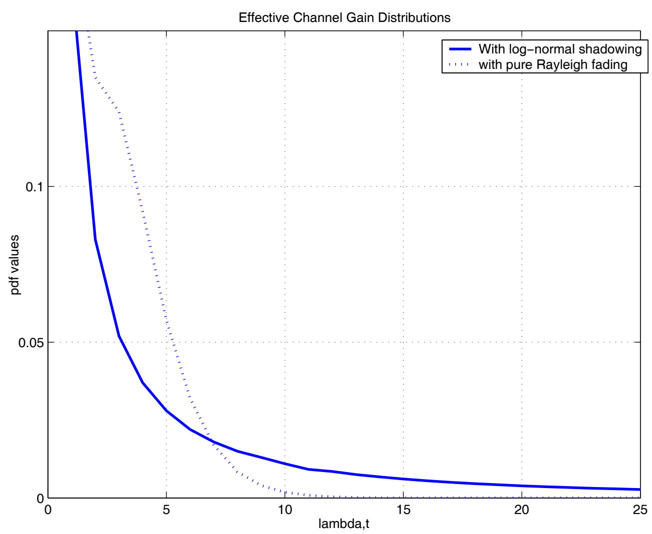

Fig. 3. Normalized channel gain for fading alone and fading plus shadowing cases.

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