International Symposium on Information Theory and its Applications, ISITA2004
Parma, Italy, October 10–13, 2004

Capacity of MIMO Systems in Rayleigh Fading with Sub-Optimal Adaptive Transmission Schemes

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Abstract

Closed form expressions for the capacity of multiple antenna systems in the presence of Rayleigh flat fading is derived for two sub-optimal adaptive transmission schemes, namely channel inversion (CI) and truncated channel inversion (TCI), assuming channel state information (CSI) is available at both the transmitter and the receiver. Moreover, an upper bound for the capacity of a multiple-input multiple-output (MIMO) system with only receiver-side channel state information (Rx-CSI) is also presented. Our results suggest that the difference in capacity between the optimal and these sub-optimal schemes reduces as the number of diversity branches increases. Further, in some cases, the capacity with sub-optimal adaptive schemes can be greater than that of a Rx-CSI only system.

1. INTRODUCTION

The capacity of fading channels varies depending on the assumptions one makes about fading statistics and the knowledge of fading coefficients. For example, [1] considered the capacity of multiple-input multiple-output (MIMO) systems when channel state information (CSI) is available only at the receiver. In this paper, we consider the capacity of multiple antenna systems with sub-optimal adaptive transmission schemes, with CSI available at both the transmitter and the receiver. This analysis has been previously considered for single antenna systems by Goldsmith and Varaiya in [2]. These adaptation techniques were later applied to single-input-multiple-output (SIMO) systems with various diversity combining techniques by Alouini and Goldsmith [3] and to multiple-input-single-output (MISO) and MIMO systems by Jayaweera and Poor [4].

In this paper, we first obtain closed form capacity expressions for MIMO systems under channel inversion (CI) and truncated channel inversion (TCI). Then, we derive the capacity expression for the special case of a MISO system and observe that this capacity is the same as that of a SIMO system employing maximal ratio combining (MRC) at the receiver as shown in [3]. This can be attributed to the principle of reciprocity which is valid when CSI is available at both transmitter and receiver. From our numerical results, it can be observed that as the number of diversity branches increases the capacity of these sub-optimal schemes approach that of the optimal scheme. Next, we provide an upper bound for the capacity of a receiver-side channel state information (Rx-CSI) only MIMO system whose exact capacity expression was obtained in [1] as an integral. In the sequel, we have also provided a direct method of evaluating moments of unordered eigenvalues of a central Wishart matrix that is instrumental in evaluating the above upper bound.

The rest of this paper is organized as follows. In section 2, we present our system model and assumptions. In section 3, we obtain the capacity of a MIMO system with the sub-optimal adaptive transmission schemes, from which we obtain the capacity for the special case of a MISO system. Also, we obtain an upper bound for the capacity of a Rx-CSI only system. Section 4 presents numerical results and section 5 contains conclusions. The derivation of the $l-th$ moment of an unordered eigenvalue of a central Wishart matrix is given in the Appendix.

2. SYSTEM MODEL DESCRIPTION

We consider a single user, flat fading communications link in which the transmitter and receiver are equipped with N_T and N_R antennas respectively. The discrete-time received signal in such a system can be written as

$$\mathbf{y}(i) \quad = \quad \mathbf{H}(i)\mathbf{x}(i) + \mathbf{n}(i) \qquad (1)$$

where $\mathbf{y(i)}$ is the $N_R \times 1$ received signal vector and $\mathbf{x}(i)$ is the $N_T \times 1$ transmitted signal vector. The matrix

$\mathbf{H}(i)$ in (1) is the $N_R \times N_T$ matrix of complex fading coefficients which are assumed to be ergodic and stationary. We assume the elements of the matrix $\mathbf{H}(i)$ are independent and identically distributed (i.i.d) circularly symmetric complex Gaussian random variables with zero mean and a variance of 1/2 per dimension. This gives rise to a Rayleigh fading channel model. The $\mathbf{n}(i)$ is the $N_R \times 1$ receiver noise vector at time i. Also, $E\left\{\mathbf{n}(i)\mathbf{n}(i)^H\right\} = N_0\mathbf{I}_{N_R}$, where $\mathbf{I}_{N_R}$ denotes the $N_R \times N_R$ identity matrix.

We assume that the CSI is available at both the transmitter and receiver and that the transmitter is subjected to an average power constraint; i.e., $\mathbb{E}\left\{\mathbf{x}^H\mathbf{x}\right\} = P$, where P is the total transmitter power. It will be shown shortly that the capacity is dependent on the transmitter and receiver antennas only through the relative parameters $n = \max\{N_R, N_T\}$ and $m = \min\{N_R, N_T\}$. Also, in this paper the base of the logarithm is taken to be two and hence capacity is expressed in bits.

3. CAPACITY OF MIMO SYSTEMS WITH SUB-OPTIMAL ADAPTIVE TRANSMISSION SCHEMES

3.1. Capacity of MIMO Systems with Channel Inversion

We start by considering the capacity of a MIMO system with channel inversion. In general, we may decompose the matrix $\mathbf{H}$ using singular value decomposition (SVD) and write the received vector as in [4] to be

$$\widetilde{\mathbf{y}} = \mathbf{\Lambda}\widetilde{\mathbf{x}} + \widetilde{\mathbf{n}}. \tag{2}$$

where $\widetilde{\mathbf{y}}$, $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{n}}$ are unitary transformations of $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{n}$ respectively. $\mathbf{\Lambda}$ is a diagonal matrix that contains the non-negative square roots of eigen values of either $\mathbf{H}\mathbf{H}^H$ or $\mathbf{H}^H\mathbf{H}$. Also, the average power constraint can be re-written as $\mathrm{tr}\left\{\widetilde{\mathbf{Q}}\right\} = P$, where $\widetilde{\mathbf{Q}} = \mathbb{E}\left\{\widetilde{\mathbf{x}}\widetilde{\mathbf{x}}^H\right\}$. As in [4], let us define

$$\mathbf{\Lambda}'(i) = \sqrt{\frac{P}{mN_0}}\mathbf{\Lambda}(i). \tag{3}$$

Also, let us define the following $m \times m$ matrix,

$$\mathbf{W} = \begin{cases} \mathbf{H}\mathbf{H}^H \text{ if } N_R \le N_T \\ \mathbf{H}^H\mathbf{H} \text{ if } N_R > N_T \end{cases}. \tag{4}$$

The distribution of $\mathbf{W}$ in (4) is given by the well-known complex central Wishart distribution.

Using the transformed model in (2) and using the definition in (3), we define the average capacity of the vector, time-varying channel with adaptive transmission scheme as in [4] to be

$$C = \max_{\widetilde{\mathbf{Q}}(\mathbf{\Lambda}')>0,\ \mathrm{tr}\left(\mathbb{E}\left\{\widetilde{\mathbf{Q}}(\mathbf{\Lambda}')\right\}\right)=P} \mathbb{E}_{\mathbf{\Lambda}'}\left\{\log_2 \det\left(\mathbf{I} + \frac{\mathbf{\Lambda}'\widetilde{\mathbf{Q}}(\mathbf{\Lambda}')\mathbf{\Lambda}'}{(P/m)}\right)\right\}. \tag{5}$$

Note that, here we assume that the channel variations are much slower compared to the data rate.Thus, the channel coefficients stay constant over a long block of symbols and then changes from block to block independently. The capacity given in (5) generalizes the definition used in [2] and [3] to a MIMO channel.

It can be shown that maximization in (5) is achieved by a diagonal $\widetilde{\mathbf{Q}}(\mathbf{\Lambda}')$. Thus, we obtain

$$C = \sum_{i=1}^{m} \mathbb{E}\left[\log_2\left(1 + \frac{\widetilde{\mathbf{Q}}_{ii}}{(P/m)}\gamma_i\right)\right], \tag{6}$$

where we define, $\gamma_i = \overline{\gamma}\lambda_i$ is the received signal-to-noise ratio (SNR), $\overline{\gamma} = \frac{P}{mN_0}$ is the average received SNR and $\widetilde{\mathbf{Q}}_{ii}$ is the (i,i)-th diagonal element of the matrix $\widetilde{\mathbf{Q}}$. Note that λ_i corresponds to the i-th eigenvalue of the central Wishart matrix defined in (4).

The probability density function (pdf) $p_\lambda(\lambda)$ of an unordered eigenvalue of a central Wishart distributed matrix was given in [1], and can be written as

$$p_\lambda(\lambda) = \frac{e^{-\lambda}\lambda^{n-m}}{m}\sum_{k=1}^{m}\frac{(k-1)!}{(n-m+k-1)!}\left[L_{k-1}^{n-m}(\lambda)\right]^2, \tag{7}$$

where the associated Laguerre polynomial of order k, L_k^{n-m}, for $k \ge 0$, is defined by [5],

$$L_k^a(\lambda) = \frac{1}{k!}e^{\lambda}\lambda^{-a}\frac{d^k}{d\lambda^k}\left[e^{\lambda}\lambda^{a+k}\right]. \tag{8}$$

If we let $f_\gamma(\gamma)$ denote the pdf of any unordered γ_i, for $i = 1, 2, \cdots, m$, then we have,

$$f_\gamma(\gamma) = \frac{1}{\overline{\gamma}}p_\lambda\left(\frac{\gamma}{\overline{\gamma}}\right). \tag{9}$$

In the case of CI, the transmitter uses instantaneous CSI to keep the received SNR per eigen-mode constant: i.e.,

$$\frac{\widetilde{\mathbf{Q}}_{ii}\gamma_i}{P/m} = \sigma.$$

Using the power constraint we can show that,

$$\sigma = \frac{1}{\mathbb{E}_\gamma\left[\frac{1}{\gamma}\right]}.$$

Using equation 7.414.12, applying a transformation formula for a hypergeometric function (equation 9.134.3) and using the definition of hypergeometric series (equation 9.100) in [5] it is straightforward to show that $\sigma = \frac{1}{\overline{\gamma}(n-m)}$, for the case of CI. Thus, we can simplify (6) and obtain capacity of a MIMO system with CI to be

$$C_{MIMO}^{CI} = m \log_2 \left(1 + \overline{\gamma}\left(n-m\right)\right) \text{ bits/channel use.} \tag{10}$$

Note that, since $m = 1$ when $N_R = 1$ (which corresponds to that of a MISO system), the capacity expression (10) reduces to

$$C_{MISO}^{CI} = \log_2 \left(1 + \overline{\gamma}\left(n-1\right)\right) \text{ bits/channel use }, \tag{11}$$

which is the same as the capacity of a SIMO system with CI and receiver maximal-ratio combining as derived in [3].

3.2. Capacity of MIMO Systems with Truncated Channel Inversion

There can be a large capacity penalty with the above channel inversion method, especially when only a small number of diversity branches are available. In those cases, we may instead consider the truncated channel inversion. Here, channel inversion is performed only when the SNR γ is above a certain threshold γ_0. i.e.,

$$P\left(\gamma\right) = \begin{cases} \frac{\sigma}{\gamma}P, & \gamma \geq \gamma_0 \\ 0, & \gamma < \gamma_0 \end{cases} .$$

Using the polynomial representation of a Laguerre polynomial (equation 8.970.1 of [5]) followed by the definition of complementary incomplete gamma function (equation 8.350.2 of [5]) we can show σ for the case of TCI to be

$$\sigma = \frac{m\overline{\gamma}}{\mathrm{A}} , \tag{12}$$

where

$$\mathrm{A} = \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!} \binom{n-m+k-1}{k-1-p}\binom{n-m+k-1}{k-1-q} \Gamma\left(n-m+p+q, \frac{\gamma_0}{\overline{\gamma}}\right) , \tag{13}$$

and the complementary incomplete gamma function is defined as $\Gamma\left(a,x\right) = \int_x^\infty e^{-t}t^{a-1}dt$.

Then capacity in this case is given by,

$$C_{MIMO}^{TCI} = \log_2\left(1+\sigma_{TCI}\right) \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \sum_{p=0}^{k-1}\sum_{q=0}^{k-1} \frac{(-1)^{p+q}}{p!q!} \binom{n-m+k-1}{k-1-p}\binom{n-m+k-1}{k-1-q} \Gamma\left(n-m+p+q+1, \frac{\gamma_0}{\overline{\gamma}}\right) . \tag{14}$$

Again, for $N_R = 1$, the capacity of a MISO system with TCI simplifies to

$$C_{MISO}^{TCI} = \log_2 \left(1 + \frac{(n-1)!\overline{\gamma}}{\Gamma\left(n-1, \frac{\gamma_0}{\overline{\gamma}}\right)}\right) \frac{\Gamma\left(n, \frac{\gamma_0}{\overline{\gamma}}\right)}{(n-1)!} , \tag{15}$$

which is the same as capacity of a SIMO system with TCI and receiver MRC as derived in [3].

3.3. Upper Bound for Rx-CSI only MIMO Systems

In this sub-section we present an upper bound for the capacity of a Rx-CSI only system for which an integral expression was derived in [1]. Since logarithm is a concave function, using the Jensen's inequality we can write

$$C_{Rx-CSI} \leq m \log_2 \left(1 + \frac{P}{N_0 N_T}\mathbb{E}_\lambda\left\{\lambda\right\}\right) . \tag{16}$$

The $l-th$ moment of an unordered eigenvalue of a central Wishart matrix can be shown to be (See Appendix)

$$\mathbb{E}_\lambda\left\{\lambda^l\right\} = \frac{1}{m}\sum_{k=1}^{m}(n-m+l)!\left[\frac{1}{(n-m)!} + \frac{\binom{l+1}{1}\binom{k-1}{1}\binom{l}{1}}{(n-m+1)!} + \frac{\binom{l+2}{2}\binom{k-1}{2}\binom{l}{2}2!}{(n-m+2)!} + \cdots + \frac{\binom{2l}{l}\binom{k-1}{l}\binom{l}{l}l!}{(n-m+l)!}\right] . \tag{17}$$

From (17), when $l = 1$,

$$\mathbb{E}_\lambda\left\{\lambda\right\} = n . \tag{18}$$

From (18) and (16) the upper bound for the capacity of a Rx-CSI only system can be written as

$$C_{Rx-CSI} \leq m \log_2 \left(1 + \frac{P}{N_0 N_T} n\right) . \tag{19}$$

It is worth recalling that, Telatar showed in [1] that for fixed receive antennas and increasing transmit antennas (i.e. N_R fixed and $N_T \to \infty$) the asymptotic capacity is given by

$$C_{Rx-CSI}^{N_T \to \infty} = N_R \log_2 \left(1 + \frac{P}{N_0}\right). \tag{20}$$

4. NUMERICAL RESULTS

Figures (1) and (2) show the capacity of a MIMO system with receive antennas fixed at $N_R = 4$ and increasing transmit antennas. Figure (1) compares the capacity of CI and TCI with the optimal adaptation scheme used in [4]. Note that this optimal adaptation scheme is a space-time water filling algorithm. As the results indicate, with the increase in number of transmit antennas the difference in capacity between optimal adaptation and that of CI or TCI reduces. Also, for large values of N_T, CI and TCI have almost the same capacity. Figure (2) compares the capacity of CI and TCI with Rx-CSI only system. For $N_T = 4$, Rx-CSI only system has a greater capacity than CI and TCI, but as the number of diversity branches increases the capacity of both sub-optimal schemes is greater than that of the Rx-CSI only system. Fig. (2) also shows the derived upper bound for a Rx-CSI only system for $N_T = 12$. Note that, for all numerical calculations involving TCI we have set $\frac{\gamma_0}{\bar{\gamma}} = 0.5$.

5. CONCLUSIONS

We considered the capacity of multiple-antenna systems in Rayleigh flat fading with sub-optimal adaptive transmission schemes, namely channel inversion and truncated channel inversion. We first derived the capacity of a MIMO system with these sub-optimal adaptive transmission schemes, using which we obtained the capacity of a MISO system as a special case. We also derived an upper bound for the capacity of a Rx-CSI only MIMO system and derived the $l-th$ moment of an unordered eigenvalue of a central Wishart matrix in general. Our numerical results suggest that for the case of fixed receive antennas and increasing number of transmit antennas, the sub-optimal schemes have greater capacity than the Rx-CSI only system for large values of N_T. Also, as the number of diversity branches increases the difference in capacity between optimal space-time water filling and the two sub-optimal schemes decreases.

Appendix

Moments of any Unordered Eigenvalue of a Central Wishart Matrix

Using the pdf of any unordered eigenvalue of a central Wishart matrix in (7), we have

$$\mathbb{E}_\lambda\left\{\lambda^l\right\} = \int_0^\infty \lambda^l \, \frac{e^{-\lambda}\lambda^{n-m}}{m} \sum_{k=1}^{m} \frac{(k-1)!}{(n-m+k-1)!} \left[L_{k-1}^{(n-m)}(\lambda)\right]^2 d\lambda \tag{21}$$

Identifying (21) with equation 7.414.12 in [5] followed by the application of a series of transformation formula for hypergeometric functions (equations 9.134.2 and 9.131.1 of [5] respectively) we get,

$$\mathbb{E}_\lambda\left\{\lambda^l\right\} = \sum_{k=1}^{m} \frac{1}{m} \frac{(n-m+l)!}{(k-1)!\ (n-m)!} \left\{ \frac{d^{k-1}}{dh^{k-1}} \left[\frac{F\left(l+1, -l; n-m+1; \frac{h}{h-1}\right)}{(1-h)} \right] \right\}_{h=0}, \tag{22}$$

where $F(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function defined as

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha.\beta}{\gamma.1} z + \frac{\alpha(\alpha+1)\ \beta(\beta+1)}{\gamma(\gamma+1).1.2} z^2 + \cdots\cdots \tag{23}$$

Upon applying the definition of Gauss hypergeometric function in (23) to (22) we obtain (17).

Acknowledgments

This research was supported in part by Kansas NASA EPSCoR program under grant KUCR #FED33144/KAN33145 and in part by a National Institute of Aviation Research (NIAR) fellowship.

References

[1] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *AT&T-Bell Labs Internal Tech. memo.,* June 1995.

[2] A. J. Goldsmith, and P. P. Varaiya, "Capacity of fading channels with channel side information," *IEEE Trans. Inform. Theory.,* vol. 43, pp. 1986-1992, Nov. 1997.

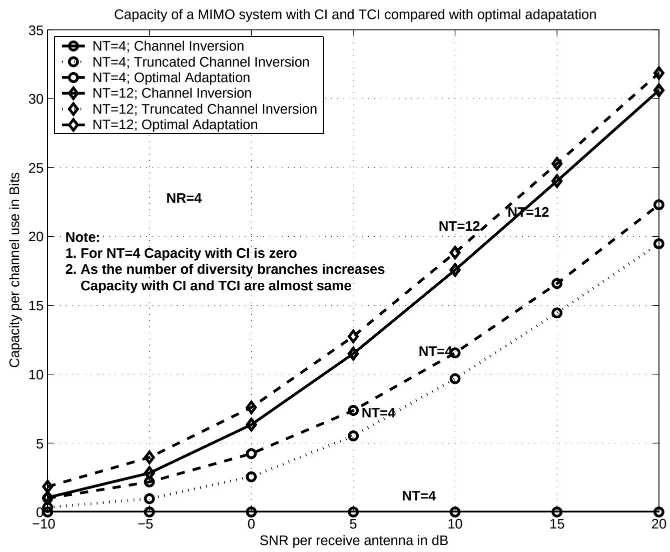

Figure 1: Comparison of Capacity of a MIMO system with the proposed sub-optimal schemes with optimal adaptation.

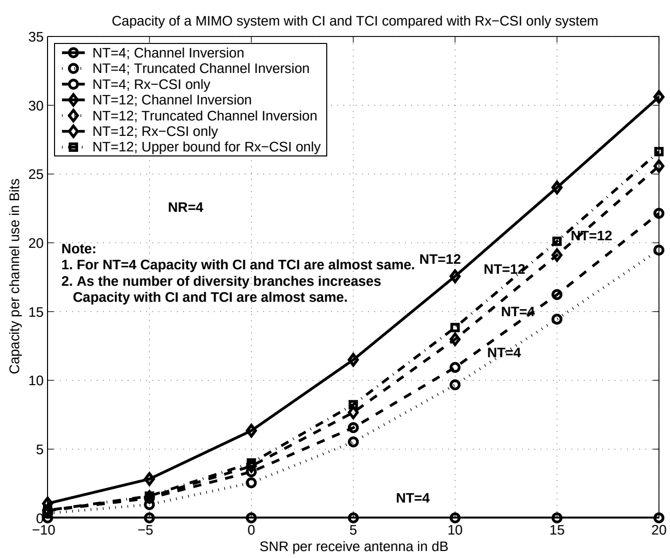

Figure 2: Comparison of Capacity of a MIMO system with the proposed sub-optimal schemes with Rx-CSI only system.

[3] M. S. Alouini, and A. J. Goldsmith, "Capacity of Rayleigh fading channels under different adaptive transmission and diversity-combining techniques," *IEEE Trans. Veh. Technol.,* vol. 48, pp. 1165-1181, July 1999.

[4] S. K. Jayaweera, and H. V. Poor, "Capacity of multiple-antenna systems with both receiver and transmitter channel state information," *IEEE Trans. Inform. Theory.,* vol. 49, pp. 2697-2709, Oct. 2003.

[5] I. S. Gradshteyn, and I. M. Ryzhik, "Table of Integrals, series, and products," Academic Press., New York, NY, USA, 1965.